Geoffrey Grimmett

Probability on Graphs

Lecture Notes on Stochastic Processes
on Graphs and Lattices

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2000 MSC: (Primary) 60K35, 82B20, (Secondary) 05C80, 82B43, 82C22

With 42 Figures

Preface

Within the menagerie of objects studied in contemporary probability theory, there are a number of related animals that have attracted great interest amongst probabilists and physicists in recent years. The overall target of these notes is to identify some of these, and to develop their basic theory. The inspiration for many of these objects comes from physics, but the mathematical subject has taken on a life of its own, and many beautiful constructions have emerged. If the two principal characters in these notes are random walk and percolation, they are only part of the rich theory of uniform spanning trees, self-avoiding walks, random networks, models for ferromagnetism and the spread of disease, and motion in random environments. This is an area that has attracted many fine scientists, by virtue, perhaps, of its special mixture of modelling and problem-solving. There remain many open problems. It is the experience of the author that these may be explained successfully to a graduate audience open to inspiration and provocation.

The material described here may be used as the basis of lecture courses of between 24 and 48 hours duration. Little is assumed about the mathematical background of the audience beyond some basic probability theory, but listeners should be willing to get their hands dirty if they are to profit. Care should be taken in the setting of examinations, since problems can be unexpectedly difficult. Successful examinations may be designed, and some help is offered through the inclusion of exercises at the ends of chapters. As an alternative to a conventional examination, students may be asked to deliver presentations on aspects and extensions of the work.

Chapter 1 is devoted to the relationship between random walks (on graphs) and electrical networks. This leads to the Thomson and Rayleigh principles, and thence to the proof of Pólya's theorem. In Chapter 2, we describe Wilson's algorithm for constructing a uniform spanning tree (UST), and we discuss boundary conditions and weak limits for UST on a lattice. This chapter includes a brief introduction to Schramm–Löwner evolutions (SLE).

Percolation theory appears first in Chapter 3, together with a short introduction to self-avoiding walks. Correlation inequalities and other general techniques are described in Chapter 4. A special feature of this part of the book is a fairly full treatment of influence and sharp-threshold theorems.

We return to percolation in Chapter 5, where the basic theory is summarised. There is a full account of Smirnov's proof of Cardy's formula. This is followed in Chapter 6 by a study of the contact model on lattices and trees.

Chapter 7 begins with a proof of the equivalence of Gibbs states and Markov fields, and continues with an introduction to the Ising and Potts models. Chapter 8 is an account of the random-cluster model. The quantum Ising model features in the next chapter, particularly through its relationship to a continuum random-cluster model, and the consequent analysis using stochastic geometry.

Interacting particle systems form the basis of Chapter 10. This is a large field in its own right, and little is done here beyond introductions to the contact, voter, and exclusion models. The chromatic number of a random graph features in Chapter 11 as an application of Hoeffding's inequality for the tail of a martingale.

The final Chapter 12 contains that most notorious open problem in stochastic geometry, the Lorentz model (or Ehrenfest wind–tree model) on the square lattice.

These notes are based in part on courses given by the author within Part 3 of the Mathematical Tripos at Cambridge University over several years, and have been prepared in this form for the 2008 PIMS–UBC Summer School in Probability, and for the programme on Statistical Mechanics at the Institut Henri Poincaré, Paris, during the last quarter of 2008. They have been written in part during a visit to the Mathematics Department at UCLA, to which the author expresses his gratitude for the warm welcome received there, and in part during programmes at the Isaac Newton Institute and the Institut Henri Poincaré–Centre Emile Borel.

The author thanks four artists for permission to include their work: Tom Kennedy (Fig. 2.1), Oded Schramm (Figs 2.2–2.4), Raphaël Cerf (Fig. 5.3), and Julien Dubédat (Fig. 5.19). The section on influence has benefited from a conversation with Rob van den Berg. Stanislav Smirnov and Wendelin Werner have consented to the inclusion of some of their neat arguments, hitherto unpublished. Several readers have proposed suggestions and corrections. Thank you, everyone!

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Cambridge
January 2009

Contents

1

Random Walks on Graphs

The theory of electrical networks is a fundamental tool for studying the recurrence of reversible Markov chains. The Kirchhoff laws and Thomson principle permit a neat proof of Pólya's theorem for random walk on a d-dimensional grid.

1.1 Random walks and reversible Markov chains

Let $G = (V, E)$ be a finite or countably infinite graph, which we assume for simplicity to have neither loops nor multiple edges. If G is infinite, we shall usually assume in addition that every vertex-degree is finite. A particle moves around the vertex-set V. Having arrived at the vertex S_n at time n, its next position S_{n+1} is chosen uniformly at random from the set of neighbours of S_n. The trajectory of the particle is called a *simple random walk* (SRW) on G.

Two of the basic questions concerning simple random walk are:

1. under what conditions is the walk *recurrent*, in that it returns (almost surely) to its starting point?
2. how does the distance between S_n and S_0 behave as $n \to \infty$?

The above SRW is symmetric in that the jumps are chosen *uniformly* from the set of available neighbours. In a more general process, we take a function $w : E \to (0, \infty)$, and we jump along the edge e with probability proportional to w_e.

Any reversible Markov chain on the set V gives rise to such a walk as follows. Let $Z = (Z_n : n \geq 0)$ be a Markov chain on V with transition matrix P, and assume that Z is reversible with respect to some positive function $\pi : V \to (0, \infty)$, which is to say that

$$\pi_u p_{u,v} = \pi_v p_{v,u}, \qquad u, v \in V. \tag{1.1}$$

With each distinct pair $u, v \in V$, we associate the weight

$$w_{u,v} = \pi_u p_{u,v}, \tag{1.2}$$

noting by (1.1) that $w_{u,v} = w_{v,u}$. Then

$$p_{u,v} = \frac{w_{u,v}}{W_u}, \qquad u, v \in V, \tag{1.3}$$

where

$$W_u = \sum_{v \in V} w_{u,v}, \qquad u \in V.$$

That is, given that $Z_n = u$, the chain jumps to a new vertex v with probability proportional to $w_{u,v}$. This may be set in the context of a random walk on the graph with the vertex-set V, and with edge-set containing all $e = \langle u, v \rangle$ such that $p_{u,v} > 0$. With the edge e we associate the weight $w_e = w_{u,v}$.

In this chapter, we develop the relationship between random walks on G and electrical networks on G. There are some excellent accounts of this area already, and the reader is referred to the books of Doyle and Snell [66] and Lyons and Peres [157], amongst others. The connection between these two topics is made via the so-called 'harmonic functions' of the random walk.

(1.4) Definition. Let $U \subseteq V$, and let Z be a Markov chain on V with transition matrix P, that is reversible with respect to the positive function π. The function $f : V \to \mathbb{R}$ is *harmonic* (for Z) on U if

$$f(u) = \sum_{v \in V} p_{u,v} f(v), \qquad u \in U,$$

or equivalently, if $f(u) = E(f(Z_1) \mid Z_0 = u)$ for $u \in U$.

From the pair (P, π), one can construct the graph G as above, and the weight function w as in (1.2). We refer to the pair (G, w) as the weighted graph associated with (P, π). We shall speak of f as being harmonic (for (G, w)) on U if it is harmonic for Z on U.

The hitting probabilities are the basic examples of harmonic functions for the chain Z. Let $W \subseteq V$ and $s \notin W$. For $u \in U = V \setminus W$, let $g(u)$ be the probability that the chain, started from u, hits s before W, that is,

$$g(u) = P_u(Z_n = s \text{ for some } n < T_W),$$

where

$$T_W = \inf\{n \geq 0 : Z_n \in W\}$$

is the first passage time to W, and $P_u(\cdot) = P(\cdot \mid Z_0 = u)$.

(1.5) Theorem. *The function g is harmonic on $U \setminus \{s\}$.*

Evidently, $g(s) = 1$, and $g(v) = 0$ for $v \in W$. We speak of these values of g as being the 'boundary conditions' of the harmonic function g.

Proof. This is an elementary exercise using the Markov property. For $u \notin W \cup \{s\}$,

$$\begin{aligned} g(u) &= \sum_{v \in V} p_{u,v} P_u(Z_n = s \text{ for some } n < T_W \mid Z_1 = v) \\ &= \sum_{v \in V} p_{u,v} g(v), \end{aligned}$$

as required. □

1.2 Electrical networks

Throughout this section, $G = (V, E)$ is a finite graph with neither loops nor multiple edges, and $w : E \to (0, \infty)$ is a weight function on the edges. We shall assume further that G is connected.

We may build an electrical network with diagram G, in which the edge e has conductance w_e (or, equivalently, resistance $1/w_e$). Let $s, t \in V$ be distinct vertices termed the *source* and *sink*, and suppose we connect a battery across the pair s, t. It is a physical observation that electrons flow along the wires in the network. The flow is described by the so-called Kirchhoff laws, as follows.

To each edge $e = \langle u, v \rangle$, there are associated (directed) quantities $\phi_{u,v}$ and $i_{u,v}$, called the *potential difference* from u to v, and the *current* from u to v, respectively. These are antisymmetric,

$$\phi_{u,v} = -\phi_{v,u}, \qquad i_{u,v} = -i_{v,u}.$$

(1.6) Kirchhoff's potential law. The cumulative potential difference around any cycle $x_1, x_2, \dots, x_n, x_{n+1} = x_1$ of G is zero, that is,

(1.7) $$\sum_{j=1}^{n} \phi_{x_j, x_{j+1}} = 0.$$

(1.8) Kirchhoff's current law. The total current flowing out of any vertex $u \in V$ other than the source and sink is zero, that is,

(1.9) $$\sum_{v \in V} i_{u,v} = 0, \qquad u \neq s, t.$$

The relationship between resistance/conductance, potential difference, and current is given by Ohm's law.

(1.10) Ohm's law. For any edge $e = \langle u, v\rangle$,

$$i_{u,v} = w_e \phi_{u,v}.$$

Kirchhoff's potential law is equivalent to the statement that there exists a function $\phi : V \to \mathbb{R}$, called a *potential function*, such that

$$\phi_{u,v} = \phi(v) - \phi(u), \qquad \langle u, v\rangle \in E.$$

Since ϕ is determined up to an additive constant, we are free to pick the potential of any single vertex. Note the convention that *current flows uphill*: $i_{u,v}$ has the same sign as $\phi_{u,v} = \phi(v) - \phi(u)$.

(1.11) Theorem. *A potential function is harmonic on the set of vertices other than the source and sink.*

Proof. Let $U = V \setminus \{s, t\}$. By Kirchhoff's current law and Ohm's law,

$$\sum_{v \in V} w_{u,v}[\phi(v) - \phi(u)] = 0, \qquad u \in U,$$

which is to say that

$$\phi(u) = \sum_{v \in V} \frac{w_{u,v}}{W_u}\phi(v), \qquad u \in U,$$

where

$$W_u = \sum_{v \in V} w_{u,v}.$$

That is, ϕ is harmonic on U. □

We can use Ohm's law to express the potential differences in terms of the currents, and thus the two Kirchhoff laws may be viewed as concerning the currents only. Relation (1.7) becomes

$$\sum_{j=1}^{n} \frac{i_{x_j,x_{j+1}}}{w_{\langle x_j,x_{j+1}\rangle}} = 0, \tag{1.12}$$

valid for any cycle $x_1, x_2, \ldots, x_n, x_{n+1} = x_1$. With (1.7) written thus, each law is linear in the currents, and the superposition principle follows.

(1.13) Theorem. Superposition principle. *If i^1 and i^2 are solutions of the two Kirchhoff laws, then so is the sum $i^1 + i^2$.*

Next we introduce the concept of a 'flow' on the graph.

(1.14) Definition. Let $s, t \in V$, $s \neq t$. An *s/t-flow* j is a vector $j = (j_{u,v} : u, v \in V,\ u \neq v)$, such that

(i) $j_{u,v} = -j_{v,u}$,

(ii) $j_{u,v} = 0$ whenever $u \nsim v$,

(iii) for any $u \neq s, t$, we have that $\sum_{v \in V} j_{u,v} = 0$.

The vertices s amd t are called the 'source' and 'sink' of an s/t-flow, and we usually abbreviate 's/t-flow' to 'flow'. For any flow j, we write

$$J_u = \sum_{v \in V} j_{u,v}, \qquad u \in U,$$

noting by (iii) above that $J_u = 0$ for $u \neq s, t$. Thus,

$$J_s + J_t = \sum_{u \in V} j_u = \sum_{u,v \in V} j_{u,v} = \tfrac{1}{2} \sum_{u,v \in V} (j_{u,v} + j_{v,u}) = 0.$$

Therefore, $J_s = -J_t$, and we call $|J_s|$ the *size* of the flow j, denoted $|j|$. If $|J_s| = 1$, we call j a *unit flow*. We shall normally take $J_s > 0$, in which case s is the source, and t the sink.

(1.15) Theorem. *Let i^1 and i^2 be two solutions of the Kirchhoff laws with equal size. Then $i^1 = i^2$.*

Proof. By the superposition principle, $j = i^1 - i^2$ satisfies the two Kirchhoff laws. Furthermore, under the flow j, no current enters or leaves the system. Therefore, $J_v = 0$ for all $v \in V$. Suppose $j_{u_1,u_2} > 0$ for some edge $\langle u_1, u_2 \rangle$. By the Kirchhoff current law, there exists u_3 such that $j_{u_2,u_3} > 0$. By iteration, there exists a cycle $u_1, u_2, \dots, u_n, u_{n+1} = u_1$ such that $j_{u_j,u_{j+1}} > 0$ for $j = 1, 2, \dots, n$. By Ohm's law, the corresponding potential function satisfies

$$\phi(u_1) < \phi(u_2) < \cdots < \phi(u_{n+1}) = \phi(u_1),$$

a contradiction. Therefore, $j_{u,v} = 0$ for all u, v. □

For a given size of input current, there can be no more than one solution to the two Kirchhoff laws, but is there a solution at all? The answer can be expressed in terms of counts of spanning trees. Consider first the special case when $w_e = 1$ for all $e \in E$. Let N be the number of spanning trees of G. For any edge $\langle a, b \rangle$, let $\Pi(s, a, b, t)$ be the property of spanning trees that: the unique s/t path in the tree passes along the edge $\langle a, b \rangle$ in the direction from a to b. Let $\mathcal{N}(s, a, b, t)$ be the set of spanning trees of G with the property $\Pi(s, a, b, t)$, and $N(s, a, b, t) = |\mathcal{N}(s, a, b, t)|$.

(1.16) Theorem. *The function*

$$i_{a,b} = \frac{1}{N}\big[N(s, a, b, t) - N(s, b, a, t)\big], \qquad \langle a, b \rangle \in E, \tag{1.17}$$

defines a flow of size 1 *that satisfies the Kirchhoff laws.*

Let T be a spanning tree of G chosen uniformly at random from the set $\mathcal{T}$ of all such spanning trees. By the theorem and the previous discussion, the unique solution to the Kirchhoff laws with size 1 is given by

$$i_{a,b} = P\big(T \text{ has } \Pi(s,a,b,t)\big) - P\big(T \text{ has } \Pi(s,b,a,t)\big).$$

We shall return to uniform spanning trees in Chapter 2.

We prove Theorem 1.16 next. Exactly the same proof is valid in the case of general conductances w_e. In that case, we define the weight of a spanning tree T as

$$w(T) = \prod_{e \in T} w_e,$$

and we set

$$(1.18) \qquad N^* = \sum_{T \in \mathcal{T}} w(T), \qquad N^*(s,a,b,t) = \sum_{T \text{ with } \Pi(s,a,b,t)} w(T).$$

The conclusion of Theorem 1.16 holds in this setting with

$$i_{a,b} = \frac{1}{N^*}\big[N^*(s,a,b,t) - N^*(s,b,a,t)\big], \qquad \langle a,b\rangle \in E,$$

Proof of Theorem 1.16. We first check the Kirchhoff current law. In every spanning tree T, there exists a unique vertex b such that the s/t path of T contains the edge $\langle s,b\rangle$, and the path traverses this edge from s to b. Therefore,

$$\sum_{b \in V} N(s,s,b,t) = N, \qquad N(s,b,s,t) = 0 \text{ for } b \in V.$$

By (1.17),

$$\sum_{b \in V} i_{s,b} = 1,$$

and, by a similar argument, $\sum_{b \in V} i_{b,t} = 1$.

Let T be a spanning tree of G. The contribution towards the quantity $i_{a,b}$, made by T, depends on the s/t path π of T, and equals

$$(1.19) \qquad \begin{array}{rl} N^{-1} & \text{if } \pi \text{ passes along } \langle a,b\rangle \text{ from } a \text{ to } b, \\ -N^{-1} & \text{if } \pi \text{ passes along } \langle a,b\rangle \text{ from } b \text{ to } a, \\ 0 & \text{if } \pi \text{ does not contain the edge } \langle a,b\rangle. \end{array}$$

Let $v \in V$, $v \neq s,t$, and write $I_v = \sum_{w \in V} i_{v,w}$. If $v \in \pi$, the contribution of T towards i_v is $N^{-1} - N^{-1} = 0$, since π arrives at v along some edge of the form $\langle a, v\rangle$, and departs v along some edge of the form $\langle v, b\rangle$. If $v \notin \pi$, then T contributes 0 to I_v. Summing over T, we obtain that $I_v = 0$ for all $v \neq s, t$, as required.

We next check the Kirchhoff potential law. Let $x_1, x_2, \dots, x_n, x_{n+1} = x_1$ be a cycle C of G. We shall show that

$$\sum_{j=1}^{n} i_{x_j, x_{j+1}} = 0, \tag{1.20}$$

and this will confirm (1.12), on recalling that $w_e = 1$ for all $e \in E$. It is more convenient in this context to work with 'bushes' than spanning trees. A *bush* is defined to be a forest on V containing exactly two trees, one denoted T_s and containing s, and the other denoted T_t and containing t. We write (T_s, T_t) for this bush. Let $e = \langle a, b\rangle$, and let $\mathcal{B}(s, a, b, t)$ be the set of bushes with $a \in T_s$ and $b \in T_t$. The sets $\mathcal{B}(s, a, b, t)$ and $\mathcal{N}(s, a, b, t)$ are in one–one correspondence, since the addition of e to a $B \in \mathcal{B}(s, a, b, t)$ creates a unique member $T = T(B)$ of $\mathcal{N}(s, a, b, t)$, and vice versa.

By (1.19) and the above, a bush $B = (T_s, T_t)$ makes a contribution to $i_{a,b}$ of:

$$\begin{array}{rl} N^{-1} & \text{if } B \in \mathcal{B}(s, a, b, t), \\ -N^{-1} & \text{if } B \in \mathcal{B}(s, b, a, t), \\ 0 & \text{otherwise.} \end{array}$$

Therefore, B makes a contribution towards the sum in (1.20) that is equal to $N^{-1}(F_+ - F_-)$, where F_+ (respectively, F_-) is the number of pairs x_j, x_{j+1} of C, $1 \le j \le n$, with $x_j \in T_s$, $x_{j+1} \in T_t$ (respectively, $x_{j+1} \in T_s$, $x_j \in T_t$). Since C is a cycle, $F_+ = F_-$, whence each bush contributes 0 to the sum, and (1.20) is proved. □

1.3 Flows and energy

Let $G = (V, E)$ be a connected graph as before, and let $s, t \in V$ be distinct vertices. Let j be an s/t-flow. With w_e the conductance of the edge e, the (dissipated) *energy* of j is defined to be

$$E(j) = \sum_{e=\langle u,v\rangle \in E} j_{u,v}^2 / w_e = \tfrac{1}{2} \sum_{\substack{u,v \in V \\ u \sim v}} j_{u,v}^2 / w_{\langle u,v\rangle}$$

The following piece of linear algebra will be useful.

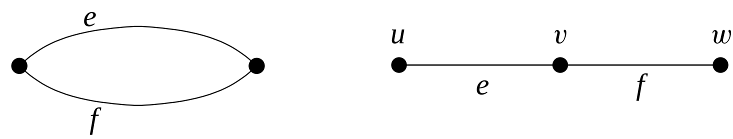

Figure 1.1. Two edges e and f in parallel and in series.

(1.21) Proposition. *Let $\psi : V \to \mathbb{R}$, and let j be an s/t-flow. Then*

$$[\psi(t) - \psi(s)]J_s = \tfrac{1}{2} \sum_{u,v \in V} [\psi(v) - \psi(u)] j_{u,v}.$$

Proof. By the properties of a flow,

$$\begin{aligned}\sum_{u,v\in V} [\psi(v) - \psi(u)] j_{u,v} &= \sum_{v \in V} \psi(v)(-J_v) - \sum_{u \in V} \psi(u) J_u \\ &= -2[\psi(s)J_s + \psi(t)J_t] \\ &= 2[\psi(t) - \psi(s)]J_s,\end{aligned}$$

as required. □

Let ϕ and i satisfy the Kirchhoff laws. We apply Proposition 1.21 with $\psi = \phi$ and $j = i$ to find by Ohm's law that

$$E(i) = [\phi(t) - \phi(s)]i_s. \tag{1.22}$$

That is, the energy of the true current-flow i between s to t equals the energy dissipated in a single $\langle s, t\rangle$ edge carrying the same potential difference and total current. The conductance W_{eff} of such an edge would satisfy Ohm's law, that is,

$$i_s = W_{\mathrm{eff}}[\phi(t) - \phi(s)], \tag{1.23}$$

and we define the *effective conductance* W_{eff} by this equation. The effective resistance is

$$R_{\mathrm{eff}} = \frac{1}{W_{\mathrm{eff}}}, \tag{1.24}$$

which, by (1.22)–(1.23), equals $E(i)/i_s^2$. We state this as a lemma.

(1.25) Lemma. *The effective resistance R_{eff} of the network between vertices s and t equals the dissipated energy when a unit flow passes from s to t.*

It is useful to be able to do calculations. Electrical engineers have devised a variety of formulaic methods for calculating the effective resistance of a network, of which the simplest are the series and parallel laws, illustrated in Figure 1.1.

(1.26) Series law. Two resistors of size r_1 and r_2 in series may be replaced by a single resistor of size $r_1 + r_2$.

(1.27) Parallel law. Two resistors of size r_1 and r_2 in parallel may be replaced by a single resistor of size R where $R^{-1} = r_1^{-1} + r_2^{-1}$.

A third such rule, the so-called 'star–triangle transformation', may be found in Exercise 1.5.

(1.28) Theorem. Thomson principle. *Let $G = (V, E)$ be a connected graph, and w_e, $e \in E$, (strictly positive) conductances. Let $s, t \in V$, $s \neq t$. Amongst all unit flows through G from s to t, the flow that satisfies the Kirchhoff laws is the unique s/t-flow i that minimizes the dissipated energy. That is,*

$$E(i) = \inf\{E(j) : j \text{ a unit flow from } s \text{ to } t\}.$$

Proof. Let j be a unit flow from source s to sink t, and set $k = j - i$ where i is the (unique) unit-flow solution to the Kirchhoff laws. Thus, k is a flow with zero size. Now, with $e = \langle u, v\rangle$ and $r_e = 1/w_e$,

$$\begin{aligned} 2E(j) &= \sum_{u,v\in V} j_{u,v}^2 r_e = \sum_{u,v\in V} (k_{u,v} + i_{u,v})^2 r_e \\ &= \sum_{u,v\in V} k_{u,v}^2 r_e + \sum_{u,v\in V} i_{u,v}^2 r_e + 2 \sum_{u,v\in V} i_{u,v} k_{u,v} r_e. \end{aligned}$$

Let ϕ be the potential function corresponding to i. By Ohm's law and Proposition 1.21,

$$\begin{aligned} \sum_{u,v\in V} i_{u,v} k_{u,v} r_e &= \sum_{u,v\in V} [\phi(v) - \phi(u)] k_{u,v} \\ &= 2[\phi(t) - \phi(s)] K_s, \end{aligned}$$

which equals zero. Therefore $E(j) \geq E(i)$, with equality if and only if $j = i$. □

The Thomson 'variational principle' leads to a proof of the 'obvious' fact that the effective resistance of a network is a non-decreasing function of the resistances of individual edges.

(1.29) Theorem. Rayleigh principle. *The effective resistance R_{eff} of the network is a non-decreasing function of the edge-resistances $(r_e : e \in E)$.*

It is left as an exercise to show that R_{eff} is a concave function of the (r_e). See Exercise 1.6.

Proof. Consider two sets $(r_e : e \in E)$ and $(r_e' : e \in E)$ of edge-resistances such that $r_e \leq r_e'$ for all e. Let i and i' denote the corresponding unit flows satisfying the Kirchhoff laws. Then, with $r_e = r_{\langle u,v\rangle}$,

$$\begin{aligned}
R_{\mathrm{eff}} &= \tfrac{1}{2} \sum_{\substack{u,v\in V\\ u\sim v}} i_{u,v}^2 r_e \\
&\le \tfrac{1}{2} \sum_{\substack{u,v\in V\\ u\sim v}} (i'_{u,v})^2 r_e && \text{by the Thomson principle}\\
&\le \tfrac{1}{2} \sum_{\substack{u,v\in V\\ u\sim v}} (i'_{u,v})^2 r'_e && \text{since } r_e \le r'_e\\
&= R'_{\mathrm{eff}},
\end{aligned}$$

as required. □

1.4 Recurrence and resistance

Let $G = (V, E)$ be an infinite connected graph with finite vertex degrees, and let $(w_e : e \in E)$ be (strictly positive) conductances. We shall consider a reversible Markov chain $Z = (Z_n : n \ge 0)$ on V with transition probabilities given by (1.3). Our purpose is to establish a condition on the pair (G, w) that is equivalent to the recurrence of Z.

Let 0 be a distinguished vertex of G, called the 'origin', and suppose $Z_0 = 0$. The graph-theoretic distance between two vertices u, v is the number of edges in the shortest path between u and v, denoted $d(u, v)$. Let

$$\begin{aligned}
\Lambda_n &= \{u \in V : d(0, v) \le n\},\\
\partial\Lambda_n = \Lambda_n \setminus \Lambda_{n-1} &= \{u \in V : d(0, v) = n\}.
\end{aligned}$$

For $n \ge 1$, we let G_n be the graph obtained from G by identifying all vertices in $V \setminus \Lambda_{n-1}$, and we denote the identified vertex as I_n. The resulting finite graph G_n may be considered as an electrical network with source 0 and sink I_n. Let $R_{\mathrm{eff}}(n)$ be the effective resistance of this network. The graph G_n may be obtained from G_{n+1} by identifying all vertices lying in $\partial\Lambda_n \cup \partial\Lambda_{n+1}$, and thus, by the Rayleigh principle, $R_{\mathrm{eff}}(n)$ is non-decreasing in n. Therefore the limit

$$R_{\mathrm{eff}} = \lim_{n\to\infty} R_{\mathrm{eff}}(n)$$

exists.

(1.30) Theorem. *The probability of return by Z to 0 is given by*

$$P_0(Z_n = 0 \text{ for some } n \ge 1) = 1 - \frac{1}{W_0 R_{\mathrm{eff}}},$$

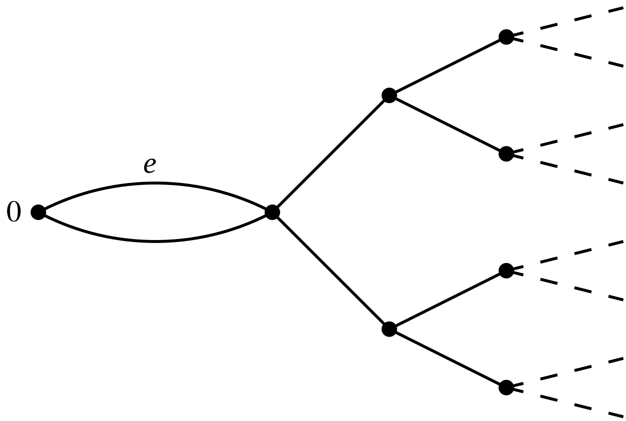

Figure 1.2. This is an infinite binary tree with two parallel edges joining the origin to the root. When each edge has unit resistance, it is an easy calculation that $R_{\text{eff}} = \frac{3}{2}$, so the probability of return to 0 is $\frac{2}{3}$. If the edge e is removed, this probability becomes $\frac{1}{2}$.

where $W_0 = \sum_{v:\, v\sim 0} w_{\langle 0,v\rangle}$.

The return probability is non-decreasing if $W_0 R_{\text{eff}}$ is increased. By the Rayleigh principle, this can be achieved, for example, by removing an edge of E that is not incident to 0. The removal of an edge incident to 0 can have the opposite effect, since W_0 decreases while R_{eff} increases. See Figure 1.2.

(1.31) Corollary.

(a) *The chain Z is recurrent if and only if $R_{\text{eff}} = \infty$.*

(b) *The chain Z is transient if and only if there exists a non-zero flow j on G from 0 to ∞ (that is, there is no sink) whose energy $E(j) = \sum_e j_e^2/w_e$ satisfies $E(j) < \infty$.*

It is left as an exercise to extend this to countable graphs G with unbounded degrees and satisfying $W_u < \infty$ for every vertex u.

Proof of Theorem 1.30. Let

$$g_n(v) = P_v(Z \text{ hits } \partial\Lambda_n \text{ before } 0), \qquad v \in \Lambda_n.$$

By Theorem 1.5, g_n is the unique harmonic function on G_n with boundary conditions

$$g_n(0) = 0, \qquad g_n(v) = 1 \text{ for } v \in \partial\Lambda_n.$$

Therefore, g_n is a potential function on G_n viewed as an electrical network with source 0 and sink I_n.

By conditioning on the first step of the walk, and using Ohm's law,

$$\begin{aligned} P_0(Z \text{ returns to 0 before reaching } \partial\Lambda_n) &= 1 - \sum_{v:\, v\sim 0} p_{0,v} g_n(v) \\ &= 1 - \sum_{v:\, v\sim 0} \frac{w_{0,v}}{W_0}[g_n(v) - g_n(0)] \\ &= 1 - \frac{|i(n)|}{W_0}, \end{aligned}$$

where $i(n)$ is the flow of currents in G_n, and $|i(n)|$ is its size. By (1.23)–(1.24), $|i(n)| = 1/R_{\text{eff}}(n)$. The theorem is proved on noting that

$$P_0(Z \text{ returns to 0 before reaching } \partial\Lambda_n) \to P_0(Z_n = 0 \text{ for some } n \ge 1)$$

as $n \to \infty$, by the continuity of probability measures. □

Proof of Corollary 1.31. Part (a) is an immediate consequence of Theorem 1.30, and we turn to part (b). By Lemma 1.25, there exists a unit flow $i(n)$ in G_n, with source 0 and sink $\partial\Lambda_n$, and with energy $E(i(n)) = R_{\text{eff}}(n)$. Let i be a non-zero $0/\infty$-flow; by normalizing by its size, we may take i to be a unit flow. When restricted to the edge-set E_n of Λ_n, i forms a unit flow from 0 to $\partial\Lambda_n$. By the Thomson principle, Theorem 1.28,

$$E(i_n) \le \sum_{e\in E_n} i_e^2/w_e \le E(i),$$

whence,

$$E(i) \ge \lim_{n\to\infty} E(i_n) = R_{\text{eff}}.$$

Therefore, by (a), $E(i) = \infty$ if the chain is transient.

Suppose, conversely, that the chain is recurrent. By diagonal selection, there exists a subsequence (n_k) along which $i(n_k)$ converges to some limit i. Since each $i(n_k)$ is a unit flow, i is a unit $0/\infty$-flow. Now,

$$\begin{aligned} E(i(n_k)) &= \sum_{e\in E} i(n_k)_e^2/w_e \\ &\ge \sum_{e\in E_m} i(n_k)_e^2/w_e \\ &\to \sum_{e\in E_m} i(e)^2/w_e \qquad \text{as } k \to \infty \\ &\to E(i) \qquad \text{as } m \to \infty. \end{aligned}$$

Therefore,

$$E(i) \le \lim_{k\to\infty} R_{\text{eff}}(n_k) = R_{\text{eff}} < \infty,$$

and i is the required flow. □

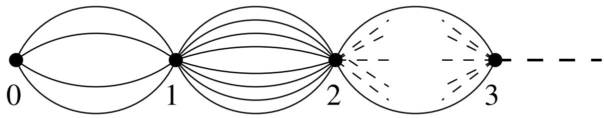

Figure 1.3. The vertex labelled i is a composite vertex obtained by identifying all vertices with distance i from 0. There are $8i - 4$ edges joining vertices $i - 1$ and i.

1.5 Pólya theorem

The following celebrated theorem[1] can be be proved by estimating effective resistances.

(1.32) Theorem [175]. *Symmetric random walk on $\mathbb{Z}^d$ is recurrent if $d = 1, 2$ and transient if $d \geq 3$.*

The advantage of the following proof of Pólya's theorem over more standard arguments is its robustness with respect to the underlying graph. Similar arguments are valid for graphs that are, in broad terms, comparable to $\mathbb{Z}^d$ when viewed as electrical networks.

Proof. For simplicity, and with only little loss of generality, we shall concentrate on the cases $d = 2, 3$. Let $d = 2$, for which case we aim to show that $R_{\text{eff}} = \infty$. This is achieved by finding an infinite lower bound for R_{eff}, and lower bounds can be obtained by decreasing individual edge-resistances. The identification of two vertices of a network amounts to the addition of a resistor with 0 resistance, and, by the Rayleigh principle, the effective resistance of the network can only decrease.

From $\mathbb{Z}^2$, we construct a new graph in which, for each $k = 1, 2, \dots$, the set $\partial\Lambda_k = \{v \in \mathbb{Z}^2 : d(0, v) = k\}$ is identified as a singleton. This transforms $\mathbb{Z}^2$ into the graph shown in Figure 1.3. By the series/parallel laws and the Rayleigh principle,

$$R_{\text{eff}}(n) \geq \sum_{i=1}^{n-1} \frac{1}{8i - 4},$$

whence $R_{\text{eff}}(n) \geq c \log n \to \infty$ as $n \to \infty$.

Suppose now that $d = 3$. There are at least two ways of proceeding. We shall present one such route from [158], and we shall then sketch the second inspired by [66]. By the remark after Theorem 1.31, it suffices to construct a non-zero flow from 0 with finite energy, and we proceed to do this. Let S be the surface of the unit sphere of $\mathbb{R}^3$ with centre at the origin 0. Take $u \in \mathbb{Z}^3$, $u \neq 0$, and position a unit cube C_u in $\mathbb{R}^3$ with centre at u; see Figure 1.4. For each neighbour v of u, the directed edge $[u, v\rangle$ intersects a unique face, denoted $F_{u,v}$, of C_u.

For $x \in \mathbb{R}^3$, $x \neq 0$, let $\Pi(x)$ be the point of intersection with S of the straight line segment from 0 to x. Let $j_{u,v}$ be equal in absolute value to the surface measure

[1] An amusing story is told in [176] about Pólya's inspiration for this theorem.

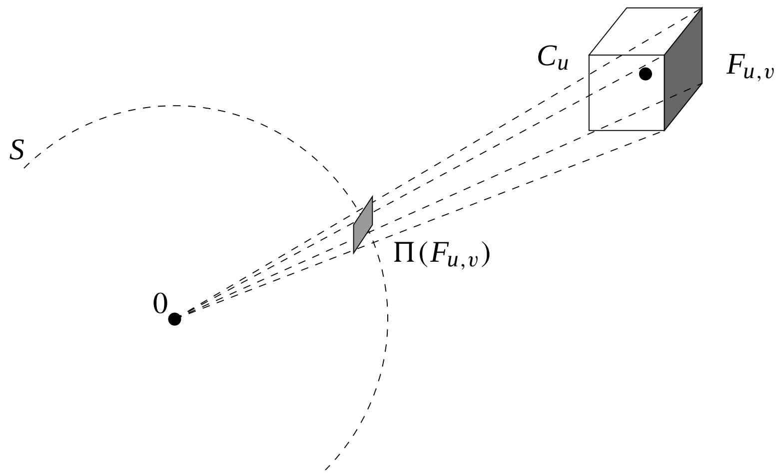

Figure 1.4. The flow along the edge $\langle u, v\rangle$ is equal to the area of the projection $\Pi(F_{u,v})$ on the unit sphere centred at the origin.

of $\Pi(F_{u,v})$. The sign of $j_{u,v}$ is taken to be positive if and only if the dot product of $\frac{1}{2}(u+v)$ and $v-u$, viewed as vectors in $\mathbb{R}^3$, is positive. Let $j_{v,u} = -j_{u,v}$. We claim that j is a flow on $\mathbb{Z}^3$. Parts (i) and (ii) of Definition 1.14 follow by construction, and it remains to check (iii).

The surface of C_u has a projection $\Pi(C_u)$ on S. The sum $J_u = \sum_{v\sim u} j_{u,v}$ is the integral over $\mathbf{x} \in \Pi(C_u)$, with respect to surface measure, of the number of neighbours v of u (counted with sign) for which $\mathbf{x} \in \Pi(F_{u,v})$. Almost every $\mathbf{x} \in \Pi(C_u)$ is counted twice, with signs $+$ and $-$. Thus the integral equals 0, whence $J_u = 0$ for all $u \neq 0$.

It is easily seen that $j_0 \neq 0$, so j is a non-zero flow. Next we estimate its energy. By an elementary geometric consideration, there exist $c_i < \infty$ such that:

(a) $|j_{u,v}| \le c_1/|u|^2$ for $u \neq 0$, where $|u| = d(0,u)$ is the length of the shortest path from 0 to u,

(b) the number of $u \in \mathbb{Z}^3$ with $|u| = n$ is smaller than $c_2 n^2$.

It follows that

$$E(j) \le \sum_{u\neq 0}\sum_{v\sim u} j_{u,v}^2 \le \sum_{n=1}^{\infty} 6c_2 n^2 \left(\frac{c_1}{n^2}\right)^2 < \infty,$$

as required. □

Another way of showing $R_{\text{eff}} < \infty$ is to find a finite upper bound for R_{eff}. Upper bounds can be obtained by increasing individual edge-resistances, or by removing edges. The idea is to embed a tree with finite resistance in $\mathbb{Z}^3$. Consider a binary tree T_ρ in which the connections between generation $n-1$ and generation n have resistance ρ^n, where $\rho > 0$. It is an easy exercise using the series/parallel laws that the effective resistance between the root and infinity is

$$R_{\text{eff}}(T_\rho) = \sum_{n=1}^{\infty} (\rho/2)^n,$$

which we make finite by choosing $\rho < 2$. We proceed to embed T_ρ in $\mathbb{Z}^3$ in such a way that a connection between generation $n-1$ and generation n is a lattice-path of length order ρ^n. There are 2^n vertices of T_ρ in generation n, and their lattice-distance from 0 has order $\sum_{k=1}^{n} \rho^k$, that is, order ρ^n. The surface of the k-ball in $\mathbb{R}^3$ has order k^2, and thus it is necessary that

$$c(\rho^n)^2 \geq 2^n,$$

which is to say that $\rho > \sqrt{2}$.

Let $\sqrt{2} < \rho < 2$. It is now fairly simple to check that $R_{\text{eff}} < c' R_{\text{eff}}(T_\rho)$. This method has been used in [102] to prove the transience of the infinite open cluster of percolation on $\mathbb{Z}^3$. It is related to, but different from, the tree embeddings of [66].

1.6 Exercises

1.1. Let $G = (V, E)$ be a finite connected graph with unit edge-weights. Show that the effective resistance between two nodes s, t of the associated electrical network may be expressed as B/N, where B is the number of bushes of G, and N is the number of its spanning trees. (See the proof of Theorem 1.16.)

Extend this result to general positive edge-weights w_e.

1.2. Let $G = (V, E)$ be a finite connected graph with positive edge-weights $(w_e : e \in E)$, and let N^* be given by (1.18). Show that

$$i_{a,b} = \frac{1}{N^*}\big[N^*(s, a, b, t) - N^*(s, b, a, t)\big]$$

constitutes a unit flow through G from s to t satisfying Kirchhoff's laws.

1.3. (continuation) Let $G = (V, E)$ be finite and connected with given conductances $(w_e : e \in E)$, and let $(x_v : v \in V)$ be reals satisfying $\sum_v x_v = 0$. To G we append a notional vertex labelled ∞, and we join ∞ to each $v \in V$. Show that there exists a solution i to Kirchhoff's laws on the expanded graph, viewed as two laws concerning current flow, such that the current along the edge $\langle v, \infty \rangle$ is x_v.

1.4. Prove the series and parallel laws for electrical networks.

1.5. *Star–triangle transformation*. The triangle T is replaced by the star S in an electrical network, as illustrated in Figure 1.5. Explain the sense in which the

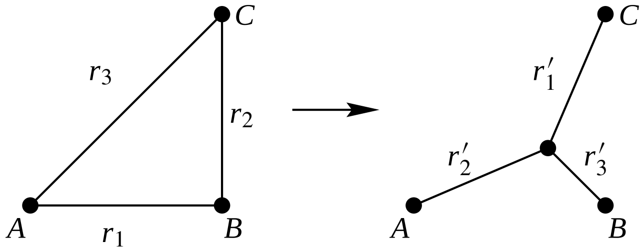

Figure 1.5. Edge-resistances in the star–triangle transformation. The triangle T on the left is replaced by the star S on the right, and the corresponding resistances are as marked.

two networks are the same, when the resistances are chosen such that $r_j r'_j = c$ for $j = 1, 2, 3$ and some constant c to be determined.

1.6. Let $R(r)$ be the effective resistance between two given vertices of a finite network with edge-resistances $r = (r(e) : e \in E)$. Show that R is concave in that

$$\tfrac{1}{2}\big[R(r_1) + R(r_2)\big] \le R\big(\tfrac{1}{2}(r_1 + r_2)\big).$$

1.7. *Maximum principle.* Let $G = (V, E)$ be a finite or infinite network with associated conductances $(w_e : e \in E)$, and let $H = (W, F)$ be a connected subgraph of G. Let $\phi : V \to [0, \infty)$ be harmonic on W, and suppose the supremum of ϕ on W is achieved and satisfies

$$\sup_{w \in W} \phi(w) = \|\phi\|_\infty := \sup_{v \in V} \phi(v).$$

Show that ϕ is constant on $W \cup \partial W$, and equals $\| f \|_\infty$.

1.8. Let G be an infinite connected graph, and let $\partial\Lambda_n$ be the set of vertices distance n from the vertex labelled 0. With E_n the number of edges joining $\partial\Lambda_n$ to $\partial\Lambda_{n+1}$, show that random walk on G is recurrent if $\sum_n E_n^{-1} = \infty$.

1.9. (continuation) Assume that G is 'spherically symmetric' in that: for all n, for all $x, y \in \partial\Lambda_n$, there exists a graph automorphism that fixes 0 and maps x to y. Show that random walk on G is transient if $\sum_n E_n^{-1} < \infty$.

1.10. Let G be a finite connected network with positive conductances $(w_e : e \in E)$, and let a, b be distinct vertices. Let i_{xy} denote the current along an edge from x to y when a unit current flows from the source vertex a to the sink vertex b. Run the associated Markov chain, starting at a, until it reaches b for the first time, and let $u_{x,y}$ be the mean of the total number of transitions of the chain between x and y. Transitions from x to y count positive, and from y to x negative, so that $u_{x,y}$ is the mean number of transitions from x to y, minus the mean number from y to x. Show that $i_{x,y} = u_{x,y}$.

1.11. Consider $\mathbb{Z}^2$ as an electrical network with unit resistances, and suppose we identify all vertices that are distance n or more from the origin. Show that the resistance between the origin and the composite vertex is at most $C \log n$ for some $C < \infty$.

2

Uniform Spanning Tree

The Uniform Spanning Tree (UST) measure has a property of negative correlation. A similar property is conjectured for Uniform Forest and Uniform Connected Subgraph. Wilson's algorithm is an efficient way to construct a UST. The UST on the infinite square grid may be defined as the weak limit of the finite-volume measures, and it converges in a certain manner to SLE_8 as the grid size approaches zero.

2.1 Definition

Let $G = (V, E)$ be a finite connected graph, and write $\mathcal{T}$ for the set of all spanning trees of G. Let T be picked uniformly at random from $\mathcal{T}$. We call T a *uniform spanning tree*, abbreviated to UST. It is governed by the uniform measure:

$$P(T = t) = \frac{1}{|\mathcal{T}|}, \qquad t \in \mathcal{T}.$$

We may think of T either as a random graph, or as a random subset of E. In the latter case, T may be thought of as a random element of the set $\Omega = \{0, 1\}^E$ of $0/1$ vectors indexed by E.

It is fundamental that UST has a property of *negative correlation*. In it simplest form, this may be expressed as follows.

(2.1) Theorem. *For* $f, g \in E$, $f \neq g$,

$$P(f \in T \mid g \in T) \le P(f \in T). \tag{2.2}$$

The proof makes striking use of the Thomson Principle via the monotonicity of effective resistance. One obtains the following by a mild extension of the proof. For $B \subseteq E$ and $g \in E \setminus B$,

$$P(B \subseteq T \mid g \in T) \le P(B \subseteq T). \tag{2.3}$$

Proof. Consider G as an electrical network each of whose edges have resistance 1. Let $e = \langle x, y\rangle$, and denote by $i = (i_{v,w} : v, w \in V)$ the current flow in G when a unit current enters at x and leaves at y. By Theorem 1.16,

$$i_{x,y} = \frac{N(x, x, y, y)}{N}$$

where $N(x, x, y, y)$ is the number of spanning trees of G whose unique x/y path passes along the edge e in the direction from x to y, and $N = |\mathcal{T}|$. Therefore, $i_{x,y} = P(e \in T)$. Since $\langle x, y\rangle$ has unit resistance, $i_{x,y}$ equals the potential difference $\phi(y) - \phi(x)$. By (1.22),

$$P(e \in T) = R^G_{\text{eff}}(x, y), \tag{2.4}$$

the effective resistance of G between x and y.

Let f, g be distinct edges, and write $G.g$ for the graph obtained from G by contracting g to a single vertex. There is a one–one correspondence between spanning trees of G containing g, and spanning trees of $G.g$. Therefore, $P(f \in T \mid g \in T)$ is simply the proportion of spanning trees of $G.g$ that contain f. By (2.4),

$$P(f \in T \mid g \in T) = R^{G.g}_{\text{eff}}(x, y).$$

By the Rayleigh principle, Theorem 1.29,

$$R^{G.g}_{\text{eff}}(x, y) \le R^G_{\text{eff}}(x, y),$$

and the theorem is proved. □

Theorem 2.1 has been extended by Feder and Mihail [78] to more general 'increasing' events. Let $\Omega = \{0, 1\}^E$, the set of 0/1 vectors indexed by E, and denote by $\omega = (\omega(e) : e \in E)$ a typical member of Ω. The partial order $\le$ on Ω is the usual pointwise ordering: $\omega \le \omega'$ if $\omega(e) \le \omega'(e)$ for all $e \in E$. A subset $A \subseteq \Omega$ is called *increasing* if: for all $\omega, \omega' \in \Omega$ satisfying $\omega \le \omega'$, we have that $\omega' \in A$ whenever $\omega \in A$.

For $A \subseteq \Omega$ and $F \subseteq E$, we say that *A is defined on F* if $A = C \times \{0, 1\}^{E\setminus F}$ for some $C \subseteq \{0, 1\}^F$. We refer to F as the 'base' of the event A.

(2.5) Theorem [78]. *Let $F \subseteq E$, and let A and B be increasing subsets of Ω such that: A is defined on F, and B is defined on $E \setminus F$. Then*

$$P(T \in A \mid T \in B) \le P(T \in A).$$

Theorem 2.1 is retrieved by setting $A = \{\omega \in \Omega : \omega(f) = 1\}$ and $B = \{\omega \in \Omega : \omega(g) = 1\}$. The original proof of Theorem 2.5 is set in the context of matroid theory, and a further proof may be found in [29].

Whereas 'positive correlation' is well developed and understood as a technique for studying interacting systems, 'negative correlation' possesses some inherent difficulties. See [173] for further discussion.

2.2 Wilson algorithm

There are various ways to generate a uniform spanning tree (UST) of the graph G. The following method, called *Wilson's algorithm* [213], highlights the close relationship between UST and random walk.

Take $G = (V, E)$ to be a finite connected graph. We shall perform random walks on G subject to a process of so-called *loop-erasure* that we describe next[1]. Let $\mathcal{W} = (w_0, w_1, \dots, w_k)$ be a walk on G, which is to say that $w_i \sim w_{i+1}$ for $0 \le i < k$ (note that the walk may have self-intersections). From $\mathcal{W}$ we construct a non-self-intersecting sub-walk, denoted $\mathrm{LE}(\mathcal{W})$, by the removal of loops as they occur. More precisely, let

$$J = \min\{j \ge 1 : w_j = w_i \text{ for some } i < j\},$$

and let I be the unique value of i satisfying $I < J$ and $w_I = w_J$. Let $\mathcal{W}' = (w_0, w_1, \dots, w_I, w_{J+1}, \dots, w_k)$ be the sub-walk of $\mathcal{W}$ obtained through the removal of the cycle $(w_I, w_{I+1}, \dots, w_J)$. This operation of single-loop-removal is iterated until no loops remain, and we denote by $\mathrm{LE}(\mathcal{W})$ the surviving path from w_0 to w_k.

Here is Wilson's algorithm. First, we order the vertex-set $V = (v_1, v_2, \dots, v_n)$ in an arbitrary manner.

1. Perform a random walk on G beginning at v_{i_1} with $i_1 = 1$, and stopped at the first time it visits v_n. The outcome is a walk $W_1 = (u_1 = v_1, u_2, \dots, u_r = v_n)$.
2. From W_1 we obtain the loop-erased path $\mathrm{LE}(W_1)$, joining v_1 to v_n and containing no loops[2]. Set $T_1 = \mathrm{LE}(W_1)$.
3. Find the earliest vertex, v_{i_2} say, of V not belonging to T_1, and perform a random walk beginning at v_{i_2}, and stopped at the first moment it hits some vertex of T_1. Call the resulting walk W_2, and loop-erase W_2 to obtain some non-self-intersecting path $\mathrm{LE}(W_2)$ from v_{i_2} to T_1. Set $T_2 = T_1 \cup \mathrm{LE}(W_2)$, the union of two edge-disjoint paths.
4. Iterate the above process, by running and loop-erasing a random walk from a new vertex $v_{i_{j+1}} \notin T_j$ until it strikes the set T_j previously constructed.
5. Stop when all vertices have been visited, and set $T = T_N$, the final value of the T_j.

Each stage of the above algorithm results in a sub-tree of G. The final such sub-tree T is spanning since, by assumption, it contains every vertex of V.

(2.6) Theorem [213]. *The graph T is a uniform spanning tree on G.*

Note that the initial ordering of V plays no role in the law of T.

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[1] Graph theorists might prefer to call this *cycle-erasure*.

[2] If we run a random walk and then erase its loops, the outcome is called *loop-erased random walk*, often abbreviated to LERW.

There are of course other ways of generating a UST on G, and we mention the well-known Aldous–Broder algorithm, [17, 48], that proceeds as follows. Choose a vertex r of G and perform a random walk on G, starting at r, until every vertex has been visited. For $w \in V$, $w \neq r$, let $[v, w\rangle$ be the directed edge that is traversed by the walk on its first visit to w. The edges thus obtained, when undirected, constitute a uniform spanning tree. The Aldous–Broder algorithm is closely related to the Wilson algorithm via a certain reversal of time, see [178].

We present the proof of Theorem 2.6 in a more general setting than UST. Heavy use will be made of [157] and the concept of 'cycle popping' introduced in the original paper [213] of David Wilson. Of considerable interest is an analysis of the run-time of Wilson's algorithm, see [178].

Consider an irreducible Markov chain with transition matrix P on the finite state space S. With this chain we may associate a directed graph $H = (S, F)$ much as in Section 1.1. This graph H has vertex-set S, and edge-set $F = \{[x, y\rangle : p_{x,y} > 0\}$. We refer to x (respectively, y) as the *head* (respectively, *tail*) of the (directed) edge $e = [x, y\rangle$, written $x = e_-$, $y = e_+$. Since the chain is irreducible, H is connected in the sense that, for all $x, y \in S$, there exists a directed path from x to y.

Let $r \in S$ be a distinguished vertex called the *root*. A *spanning arborescence* of H with root r is a subgraph A with the following properties:

(a) each vertex of S apart from r is the head of a unique edge of A,

(b) the root r is the head of no edge of A,

(c) A possesses no (directed) cycles.

Let Σ_r be the set of all spanning arborescences with root r, and $\Sigma = \bigcup_{r \in S} \Sigma_r$.

It is easily seen that there exists a unique (directed) path in the spanning arborescence A joining any given vertex x to the root. To the spanning arborescence A we assign the weight

$$\alpha(A) = \prod_{e \in A} p_{e_-,e_+}, \tag{2.7}$$

and we shall describe a randomized algorithm that selects a given spanning arborescence A with probability proportional to $\alpha(A)$. Since $\alpha(A)$ is independent of the diagonal elements $p_{z,z}$ of P, and each x ($\neq r$) is the head of a unique edge of A, we may assume that $p_{z,z} = 0$ for all $z \in S$.

Let $r \in S$. Wilson's algorithm is easily adapted in order to sample from Σ_r. Let $v_1, v_2, \dots, v_{n-1}$ be an ordering of $S \setminus \{r\}$.

1. Let $\sigma_0 = \{r\}$.
2. Sample a Markov chain with transition matrix P beginning at v_{i_1} with $i_1 = 1$, and stopped at the first time it hits σ_0. The outcome is a (directed) walk $W_1 = (u_1 = v_1, u_2, \dots, u_k, r)$. From W_1 we obtain the loop-erased path $\sigma_1 = \mathrm{LE}(W_1)$, joining v_1 to r and containing no loops.
3. Find the earliest vertex, v_{i_2} say, of S not belonging to σ_1, and sample a Markov chain beginning at v_{i_2}, and stopped at the first moment it hits some

vertex of σ_1. Call the resulting walk W_2, and loop-erase it to obtain some non-self-intersecting path $\mathrm{LE}(W_2)$ from v_{i_2} to σ_1. Set $\sigma_2 = \sigma_1 \cup \mathrm{LE}(W_2)$, the union of σ_1 and the directed path $\mathrm{LE}(W_2)$.

4. Iterate the above process, by loop-erasing the trajectory of a Markov chain starting at a new vertex $v_{i_{j+1}} \notin \sigma_j$ until it strikes the graph σ_j previously constructed.
5. Stop when all vertices have been visited, and set $\sigma = \sigma_N$, the final value of the σ_j.

(2.8) Theorem [213]. *The graph σ is a spanning arborescence with root r, and*

$$P(\sigma = A) \propto \alpha(A), \qquad A \in \Sigma_r.$$

Since S is finite and the chain is assumed irreducible, there exists a unique stationary distribution $\pi = (\pi_s : s \in S)$. Suppose that the chain is reversible with respect to π in that

$$\pi_x p_{x,y} = \pi_y p_{y,x}, \qquad x, y \in S.$$

As in Section 1.1, to each edge $e = [x, y\rangle$ we may allocate the weight $w(e) = \pi_x p_{x,y}$, noting that the edges $[x, y\rangle$ and $[y, x\rangle$ have equal weight. Let A be a spanning arborescence with root r. Since each vertex of H other than the root is the head of a unique edge of the spanning arborescence A, we have by (2.7) that

$$\alpha(A) = \frac{\prod_{e\in A} \pi_{e_-} p_{e_-,e_+}}{\prod_{x\in S,\, x\neq r} \pi_x} = CW(A), \qquad A \in \Sigma_r,$$

where $C = C_r$ and

$$W(A) = \prod_{e\in A} w(e).$$

Therefore, for a given root r, the weight functions α and W generate the same probability measure on Σ_r. The UST measure on $G = (V, E)$ arises through a consideration of the random walk on G, i.e., by taking $p_{x,y} = 1/\deg(x)$. By Theorem 2.8, Wilson's algorithm generates a random spanning arborescence σ with given root on the graph H obtained from G by duplicating and directing the edges. When we neglect the orientations of the edges of σ, and also the identity of the root, σ is transformed into a uniform spanning tree of G.

The remainder of this section is devoted to a proof of Theorem 2.8, and it uses the beautiful construction presented in [213].

For each $x \in S \setminus \{r\}$, we provide ourselves in advance with an infinite set of 'moves' from x. Let $M_x(i)$, $i \geq 1$, $x \in S \setminus \{r\}$, be independent random variables with laws

$$P(M_x(i) = y) = p_{x,y}, \qquad y \in S.$$

For each x, we organize the $M_x(i)$ into an ordered 'stack'. We think of an element $M_x(i)$ as having 'colour' i, where the colours indexed by i are distinct. The root r is given an empty stack. At stages of the following construction, we shall discard elements of stacks in order of increasing colour, and we shall call the set of uppermost elements of the stacks the 'visible moves'.

The visible moves generate a directed subgraph of H termed the 'visible graph'. There will generally be directed cycles in the visible graph, and we shall remove such cycles one by one. Whenever we decide to remove a cycle, the corresponding visible moves are removed from the stacks, and a new set of moves beneath is revealed. The visible graph thus changes, and a second cycle may be removed. This process may be iterated until the earliest time, N say, at which the visible graph contains no cycle, which is to say that the visible graph is a spanning arborescence σ with root r. If $N < \infty$, we terminate the procedure and 'output' σ. The removal of a cycle is called 'cycle popping'. It would seem that the value of N and the output σ will depend on the order in which we decide to pop cycles, but the converse turns out to be the case.

The following lemma holds 'pointwise': it contains no statement involving probabilities.

(2.9) Lemma. *The order of cycle popping is irrelevant to the outcome, in that*

either: *for all orderings of cycle popping, $N = \infty$,*

or: *the total number N of popped cycles, and the output σ, are independent of the order of popping.*

Proof. A *coloured cycle* is a sequence $M_{x_j}(i_j)$, $j = 1, 2, \dots, J$, that constitutes a cycle of H. A coloured cycle C is called *poppable* if there exists a sequence $C_1, C_2, \dots, C_n = C$ of coloured cycles that may be popped in order. We claim the following for any cycle-popping algorithm. If the algorithm terminates in finite time, then all poppable cycles are popped, and no others. The lemma follows from this claim.

Let C be a poppable coloured cycle, and let $C_1, C_2, \dots, C_n = C$ be as above. It suffices to show the following. Let $C' \neq C_1$ be a poppable cycle with colour 1, and suppose we pop C' at the first stage, rather than C_1. *Then C is still poppable after the removal of C'.*

Let $V(D)$ denote the vertex-set of a coloured cycle D. The italicized claim is evident if $V(C') \cap V(C_k) = \varnothing$ for $k = 1, 2, \dots, n$. Suppose on the contrary that $V(C') \cap V(C_k) \neq \varnothing$ for some k, and let K be the earliest such k. Let $x \in V(C') \cap V(C_K)$. Since $x \notin V(C_k)$ for $k < K$, the visible move at x has colour 1 even after the popping of $C_1, C_2, \dots, C_{K-1}$. Therefore, the edge of C_K with head x has the same tail, y say, as that of C' with head x. This argument may be applied to y also, and then to all vertices of C_K in order. In conclusion, C_K has colour 1, and $C' = C_K$.

Were we to decide to pop C' first, then we may choose to pop in the sequence C_K [$= C'$], $C_1, C_2, C_3, \dots, C_{K-1}, C_{K+1}, \dots, C_n = C$, and the claim has been shown. □

Proof of Theorem 2.8. It is clear by construction that the Wilson algorithm terminates after finite time, with probability 1. It proceeds by popping cycles, and so, by Lemma 2.9, $N < \infty$ almost surely, and the output σ is independent of the choices available in its implementation.

We show next that σ has the required law. We may think of the stacks as generating a pair $(\mathbf{C}, \sigma)$, where $\mathbf{C} = (C_1, C_2, \dots, C_J)$ is the ordered set of coloured cycles that are popped by Wilson's algorithm, and σ is the spanning arborescence thus revealed. Note that the colours of the moves of σ are determined by knowledge of $\mathbf{C}$. Let $\mathcal{C}$ be the set of all sequences $\mathbf{C}$ that may occur, and Π the set of all possible pairs $(\mathbf{C}, \sigma)$. Certainly $\Pi = \mathcal{C} \times \Sigma_r$, since knowledge of $\mathbf{C}$ imparts no information about σ.

The law of $(\mathbf{C}, \sigma)$ is simply the probability that the coloured moves are given appropriately. That is,

$$P\big((\mathbf{C}, \sigma) = (\mathbf{c}, A)\big) = \left(\prod_{c \in \mathbf{c}} \prod_{e \in c} p_{e_-, e_+}\right) \alpha(A), \qquad \mathbf{c} \in \mathcal{C},\ A \in \Sigma_r.$$

Since this factorizes in the form $f(\mathbf{c})g(A)$, the random variables $\mathbf{C}$ and σ are independent, and $P(\sigma = A)$ is proportional to $\alpha(A)$ as required. □

2.3 Weak limits on lattices

Let $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$ be the d-dimensional hypercubic lattice, with $d \ge 2$. Let μ_n be the UST measure on the box $B(n) = [-n, n]^d$.

(2.10) Theorem [170]. *The weak limit $\mu = \lim_{n\to\infty} \mu_n$ exists and is a translation-invariant and ergodic*[3] *probability measure. It is supported on the set of forests in $\mathbb{L}^d$ with no bounded component.*

Since we are working in the σ-field of Ω generated by the cylinder events, it suffices for weak convergence[4] that $\mu_n(B \subseteq T) \to \mu(B \subseteq T)$ for any finite set B of edges (see Exercise 2.3). Note that the limit measure μ may place strictly positive probability on the set of forests with two or more components. By a mild extension of the proof, one obtains that the limit measure μ is invariant under the action of any automorphism of the lattice $\mathbb{L}^d$.

Proof. Let F be a finite set of edges of $\mathbb{E}^d$. By the Rayleigh principle, Theorem 1.29 (as in the proof of Theorem 2.1, see Exercise 2.4),

$$\mu_n(F \subseteq T) \ge \mu_{n+1}(F \subseteq T),$$

[3] μ is *ergodic* if any shift-invariant event A has probability either 0 or 1.

[4] A brief note about weak convergence can be found at the end of this section.

for all large n. Therefore, the limit

$$\mu(F \subseteq T) = \lim_{n\to\infty} \mu_n(F \subseteq T)$$

exists. The domain of μ may be extended to all cylinder events, by the inclusion–exclusion principle, and this in turn specifies a unique probability measure μ on the infinite grid. Since no tree contains a cycle, and since each cycle is finite and there are countably many cycles in $\mathbb{L}^d$, μ has support in the set of forests. By a similar argument, these forests may be taken with no bounded component.

Let π be a translation of $\mathbb{Z}^2$, and let F be finite as above. Then

$$\mu(\pi F \subseteq T) = \lim_{n\to\infty} \mu_n(\pi F \subseteq T) = \lim_{n\to\infty} \mu_{\pi,n}(F \subseteq T),$$

where $\mu_{\pi,n}$ is the law of a UST on $\pi^{-1}B(n)$. There exists $r = r(\pi)$ such that $B(n-r) \subseteq \pi^{-1}B(n) \subseteq B(n+r)$ for all large n. By the Rayleigh principle again,

$$\mu_{n+r}(F \subseteq T) \le \mu_{\pi,n}(F \subseteq T) \le \mu_{n-r}(F \subseteq T)$$

for all large n. Therefore,

$$\mu_{\pi,n}(F \subseteq T) \to \mu(F \subseteq T),$$

whence the translation-invariance of μ. The proof of ergodicity is omitted, and may be found in [170]. □

This leads immediately to the question of whether or not the support of μ is the set of spanning trees of $\mathbb{L}^d$.

(2.11) Theorem [170]. *The limit measure μ is supported on the set of spanning trees of $\mathbb{L}^d$ if and only if $d \le 4$.*

The above measure μ may be termed 'free UST measure'. There is another possible boundary condition giving rise to the so-called 'wired UST measure'. One identifies as a single vertex all vertices not in $B(n-1)$, and chooses a spanning tree uniformly at random from the resulting (finite) graph. One can pass to the limit as $n \to \infty$ in very much the same way as before. It turns out that the free and wired measures are identical on $\mathbb{L}^d$ for all d. The key fact is that $\mathbb{L}^d$ is a so-called *amenable* graph, which amounts in this context to saying that the boundary/volume approaches zero in the limit of large boxes,

$$|\partial B(n)|/|B(n)| \to 0 \qquad \text{as } n \to \infty.$$

See Exercise 2.8 and [29, 157, 170, 171] for further details and discussion.

This section closes with a brief note about weak convergence, for more details of which the reader is referred to the books [36, 67]. Let $E = \{e_i : 1 \le i < \infty\}$ be a countably infinite set. The product space $\Omega = \{0, 1\}^E$ may be viewed as the product of copies of the discrete topological space $\{0, 1\}$ and, as such, Ω is compact, and is metrisable by

$$\delta(\omega, \omega') = \sum_{i=1}^{\infty} 2^{-i} |\omega(e_i) - \omega'(e_i)|, \qquad \omega, \omega' \in \Omega.$$

A subset C of Ω is called a *cylinder event* (or, simply, a cylinder) if there exists a finite $F \subseteq E$ such that: $\omega \in C$ if and only if $\omega' \in C$ for all ω' equal to ω on F. The *product σ-algebra* $\mathcal{F}$ of Ω is the σ-algebra generated by the cylinders. The *Borel σ-algebra* $\mathcal{B}$ of Ω is defined as the minimal σ-algebra containing the open sets. It is standard that $\mathcal{B}$ is generated by the cylinders, and therefore $\mathcal{F} = \mathcal{B}$. We note that every cylinder is both open and closed in the product topology.

Let $(\mu_n : n \geq 1)$ and μ be probability measures on $(\Omega, \mathcal{F})$. We say that μ_n converges *weakly* to μ, written $\mu_n \Rightarrow \mu$, if

$$\mu_n(f) \to \mu(f) \qquad \text{as } n \to \infty,$$

for all bounded continuous functions $f : \Omega \to \mathbb{R}$. (As usual, $P(f)$ denotes the expectation of the function f under the measure P.) Several other definitions of weak convergence are possible, and the so-called 'portmanteau theorem' asserts that certain of these are equivalent. In particular, the weak convergence of μ_n to μ is equivalent to each of the two following statements:

(i) $\limsup_{n\to\infty} \mu_n(C) \leq \mu(C)$ for all closed events C,

(ii) $\liminf_{n\to\infty} \mu_n(C) \geq \mu(C)$ for all open events C.

The matter is simpler in the current setting: since the cylinder events are both open and closed, and they generate $\mathcal{F}$, it is necessary and sufficient for weak convergence that

(iii) $\lim_{n\to\infty} \mu_n(C) = \mu(C)$ for all cylinders C.

The following is useful for the construction of infinite-volume measures in the theory of interacting systems. Since Ω is compact, every family of probability measures on $(\Omega, \mathcal{F})$ is relatively compact. That is to say, for any such family $\Pi = (\mu_i : i \in I)$, every sequence $(\mu_{n_k} : k \geq 1)$ in Π possesses a weakly convergent subsequence. Suppose now that $(\mu_n : n \geq 1)$ is a sequence of probability measures on $(\Omega, \mathcal{F})$. If the limits $\lim_{n\to\infty} \mu_n(C)$ exists for every cylinder C, then it is necessarily the case that $\mu := \lim_{n\to\infty} \mu_n$ exists and is a probability measure. We shall see in Exercises 2.2–2.3 that this holds if and only if $\lim_{n\to\infty} \mu_n(C)$ exists for all *increasing* cylinders C. This justifies the argument of the proof of Theorem 2.10.

2.4 Uniform Forest

We saw in Theorems 2.1 and 2.5 that the UST has a property of negative correlation. There is evidence that certain related measures have such a property also, but such claims have resisted proof.

Let $G = (V, E)$ be a finite graph, which we may as well assume to be connected. Write $\mathcal{F}$ for the set of forests of G (that is, subsets $H \subseteq E$ containing no cycles), and $\mathcal{C}$ for the set of connected subgraphs of G (that is, subsets $H \subseteq E$ such that (V, H) is connected). Let F be a uniformly chosen member of $\mathcal{F}$, and C a uniformly chosen member of $\mathcal{C}$. We refer to F and C as a *uniform forest* (UF) and a *uniform connected subgraph* (USC), respectively.

(2.12) Conjecture. *For $f, g \in E$, $f \neq g$, the UF and USC satisfy:*

$$P(f \in F \mid g \in F) \le P(f \in F), \tag{2.13}$$

$$P(f \in C \mid g \in C) \le P(f \in C). \tag{2.14}$$

One may ask whether UF and USC satisfy the stronger conclusion of Theorem 2.5. As positive evidence of Conjecture 2.12, we cite the computer-aided proof of [111] that the UF on any graph with eight or fewer vertices (or nine vertices and eighteen or fewer edges) satisfies (2.13).

Discuss general approaches to negative correlation, [131, 173].

2.5 Schramm–Löwner evolutions

There is a beautiful result of Lawler, Schramm, and Werner [147] concerning the limiting LERW (loop-erased random walk) and UST measures on $\mathbb{L}^2$. This cannot be described without a detour into the theory of Schramm–Löwner evolutions[5] (SLE).

Let $\mathbb{H} = (-\infty, \infty) \times (0, \infty)$ be the upper half-plane of $\mathbb{R}^2$, with closure $\overline{\mathbb{H}}$, viewed as subsets of the complex plane. Consider the (Löwner) ordinary differential equation

$$\frac{d}{dt} g_t(z) = \frac{2}{g_t(z) - b(t)}, \qquad z \in \overline{\mathbb{H}} \setminus \{0\},$$

subject to the boundary condition $g_0(z) = z$, where $t \in [0, \infty)$, and $b : \mathbb{R} \to \mathbb{R}$ is termed the 'driving function'. Randomness in injected into this formula through setting $b(t) = B_{\kappa t}$ where $\kappa > 0$ and $(B_t : t \ge 0)$ is a standard Brownian motion[6]. The solution exists when $g_t(z)$ is bounded away from $B_{\kappa t}$. More specifically, for $z \in \overline{\mathbb{H}}$, let τ_z be the infimum of all times τ such that 0 is a limit point of $g_s(z) - B_{\kappa s}$ in the limit as $s \uparrow \tau$. We let

$$H_t = \{z \in \mathbb{H} : \tau_z > t\}, \qquad K_t = \{z \in \overline{\mathbb{H}} : \tau_z \le t\},$$

[5] Originally known as 'stochastic Löwner evolutions, but now often renamed after Schramm, in recognition of [189].

[6] See [68] for an interesting and topical account of the history and practice of Brownian motion.

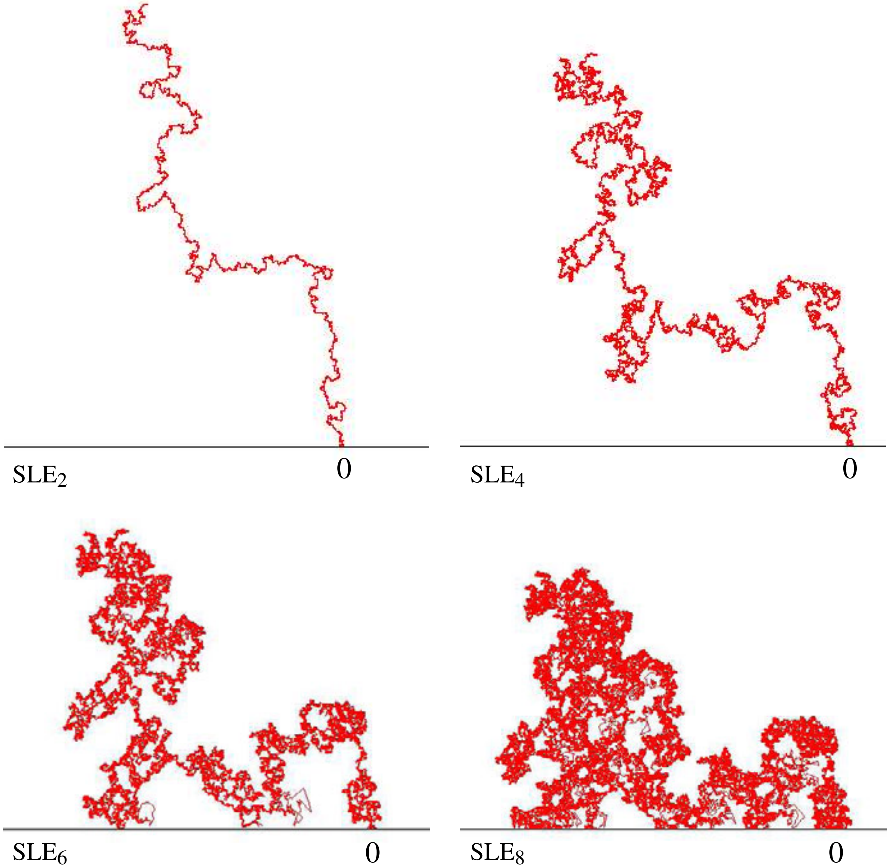

Figure 2.1. Simulations of chordal SLE_κ for $\kappa = 2, 4, 6, 8$. The four pictures are generated from the same Brownian driving path.

so that H_t is open, and K_t is compact. It may now be seen that g_t is a conformal homeomorphism from H_t to $\mathbb{H}$. The process may be described via a random curve $\gamma : [0, \infty) \to \overline{\mathbb{H}}$ in the sense that $\mathbb{H} \setminus K_t$ is the unbounded component of $\mathbb{H} \setminus \gamma[0, t]$. The curve γ satisfies $\gamma(0) = 0$ and $\gamma(t) \to \infty$ as $t \to \infty$. See the illustrations of Figure 2.1.

We call $(g_t : t \geq 0)$ a *Schramm–Löwner evolution* (SLE) with parameter κ, written SLE_κ, and we call the K_t the *hulls* of the process. There is good reason to believe that the family $K = (K_t : t \geq 0)$ provides the correct scaling limits for a variety of random spatial processes, with the value of κ depending on the process in question. General properties of SLE_κ, viewed as a function of κ, have been studied in [182, 208, 209], and a beautiful theory has emerged. For example, the hulls K form (almost surely) a simple path if and only if $\kappa \leq 4$. If $\kappa > 8$, then SLE_κ generates (almost surely) a space-filling curve.

The above SLE is termed 'chordal'. In another version, called 'radial' SLE, the upper half-plane $\mathbb{H}$ is replaced by the unit disc $\mathbb{U}$, and a different differential equation is satisfied. The corresponding curve γ satisfies $\gamma(t) \to 0$ as $t \to \infty$,

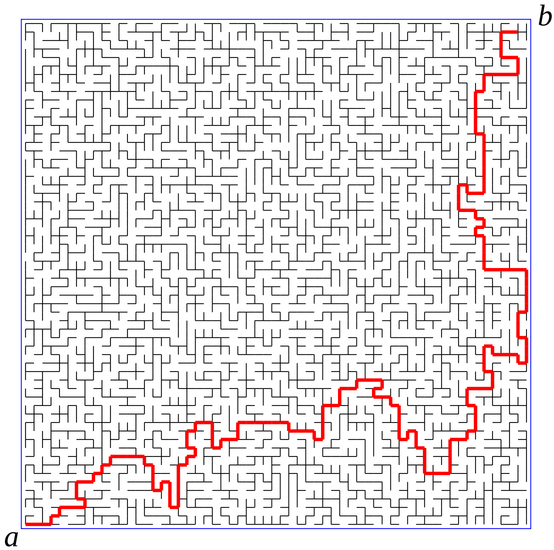

Figure 2.2. The unique UST path between opposite corners a, b of a square. It has the law of a LERW between a and b.

and $\gamma(0) \in \partial\mathbb{U}$, say $\gamma(0) = 1$. Both chordal and radial SLE may be defined on an arbitrary simply connected domain D with a boundary, by applying a suitable conformal map ϕ from either $\mathbb{H}$ or $\mathbb{U}$ to D.

Schramm [189, 190] identified the correct value of κ for several different processes, and indicated that percolation has scaling limit SLE_6. Full rigorous proofs are not yet known even for general percolation models. For the special case of site percolation on the triangular lattice $\mathbb{T}$, Smirnov [196, 197] has proved the very remarkable result that the crossing probabilities of re-scaled regions of $\mathbb{R}^2$ satisfy Cardy's formula, see Section 5.6.

The theory of SLE is a major piece of contemporary mathematics which promises to explain phase transitions in an important class of two-dimensional disordered systems, and to help bridge the gap between probability theory and conformal field theory. It has in addition provided complete explanations of conjectures made by mathematicians and physicists concerning the intersection exponents and fractionality of frontier of two-dimensional Brownian motion, see [144, 145].

This chapter closes with a brief summary of the results of [147] concerning SLE limits for LERW and UST on the square lattice $\mathbb{L}^2$. We saw earlier in this chapter that there is a very close relationship between LERW and UST on a finite connected graph G. For example, the unique path joining vertices u and v in a UST of G has the law of a LERW from u to v (see [170] and the description of Wilson's algorithm). See Figure 2.2.

Let D be a bounded simply connected subset of $\mathbb{C}$ with $0 \in D$. As remarked above, we may define radial SLE_2 on D, and we write ν for its law. Let $\delta > 0$, and let μ_δ be the law of LERW on the re-scaled lattice $\delta\mathbb{Z}^2$, starting at 0 and stopped

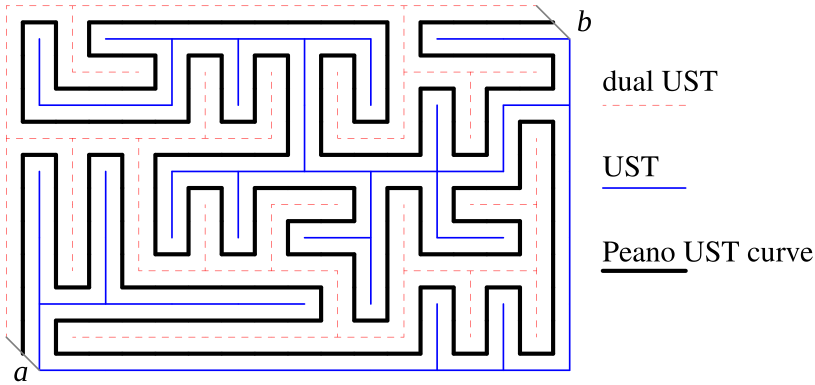

Figure 2.3. An illustration of the Peano UST path lying between a tree and its dual. The thinner continuous line depicts the UST, and the dashed line its dual tree. The thicker line is the Peano UST path.

when it first hits ∂D.

For two parametrizable curves β, γ in $\mathbb{C}$, we define the distance between them by

$$\rho(\beta, \gamma) = \inf \left[\sup_{t \in [0,1]} |\widehat{\beta}(t) - \widehat{\gamma}(t)| \right],$$

where the infimum is over all parametrizations $\widehat{\beta}$ and $\widehat{\gamma}$ of the curves (see [8]). The distance function ρ generates a topology on the space of parametrizable curves, and hence a notion of weak convergence (denoted '$\Rightarrow$').

(2.15) Theorem [147]. *We have that* $\mu_\delta \Rightarrow \nu$ *as* $\delta \to 0$.

We turn to the convergence of UST to SLE_8, and begin with a discussion of mixed boundary conditions. Let D be a bounded simply connected domain of $\mathbb{C}$ with a smooth (C^1) boundary curve ∂D. For distinct points $a, b \in \partial D$, we write α (respectively, β) for the arc of ∂D going clockwise from a to b (respectively, b to a). Let $\delta > 0$ and let G_δ be a connected graph that approximates to that part of $\delta\mathbb{Z}^2$ lying inside D. We shall construct a UST of G_δ with mixed boundary conditions, namely a free boundary near α and a wired boundary near β. To each tree T of G_δ there corresponds a dual tree T^{d} on the dual graph G_δ^{d}, namely the tree comprising edges of G_δ^{d} that do not intersect those of T. Since G_δ has mixed boundary conditions, so does its dual G_δ^{d}. With G_δ and G_δ^{d} drawn together, there is a simple path $\pi(T, T^{\mathrm{d}})$ that winds between T and T^{d}. Let Π be the path thus constructed between the UST on G_δ and its dual tree. The construction of this 'Peano UST curve' is illustrated in Figures 2.3 and 2.4.

(2.16) Theorem [147]. *The law of* Π *converges as* $\delta \to 0$ *to that of the image of chordal* SLE_8 *under any conformal map from* $\mathbb{H}$ *to* D *mapping* 0 *to* a *and* ∞ *to* b.

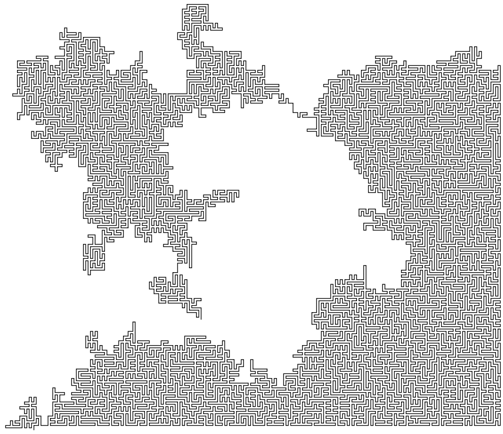

Figure 2.4. An initial segment of the Peano path constructed from a UST on a large square with mixed boundary conditions.

2.6 Exercises

2.1. [17, 48] *Aldous–Broder algorithm.* Let $G = (V, E)$ be a finite connected graph, and pick a root $r \in V$. Perform a random walk on G starting from r. For each $v \in V$, $v \neq r$, let e_v be the edge traversed by the random walk just before it hits v for the first time, and let T be the tree $\bigcup_v e_v$ rooted at r. Show that T, when viewed as an unrooted tree, is a uniform spanning tree. It may be helpful to argue as follows.

a. Consider a stationary simple random walk $(X_n : -\infty < n < \infty)$ on G, with distribution $\pi_v \propto \deg(v)$, the degree of v. Let T_i be the rooted tree obtained by the above procedure applied to the sub-walk $X_i, X_{i+1}, \dots$. Show that $T = (T_i : -\infty < i < \infty)$ is a stationary Markov chain with state space the set $\mathcal{R}$ of rooted spanning trees.

b. Let $Q(t, t') = P(T_0 = t' \mid T_1 = t)$, and let $r(t)$ be the degree of the root of $t \in \mathcal{R}$. Show that:

  (i) for given $t \in \mathcal{R}$, there are exactly $r(t)$ trees $t' \in \mathcal{R}$ with $Q(t, t') = 1/r(t)$, and $Q(t, t') = 0$ for all other t',

  (ii) for given $t' \in \mathcal{R}$, there are exactly $r(t')$ trees $t \in \mathcal{R}$ with $Q(t, t') = 1/r(t)$, and $Q(t, t') = 0$ for all other t.

c. Show that

$$\sum_{t \in \mathcal{R}} r(t) Q(t, t') = r(t'), \qquad t' \in \mathcal{R},$$

and deduce that the stationary measure of T is proportional to $r(t)$.

d. Let $r \in V$, and let t be a tree with root r. Show that $P(T_0 = t \mid X_0 = r)$ is independent of the choice of t.

2.2. Let $\Omega = \{0, 1\}^F$ where F is finite, and let P be a probability measure on

Ω, and $A \subseteq \Omega$. Show that $P(A)$ may be expressed as a linear combination of certain $P(A_i)$ where the A_i are increasing events.

2.3. (continuation) Let $G = (V, E)$ be an infinite graph with finite vertex-degrees, and $\Omega = \{0, 1\}^E$. An event A in the product σ-field of Ω is called a *cylinder event* if it has the form $A_F \times \{0, 1\}^{\overline{F}}$ for some $A_F \subseteq \{0, 1\}^F$ and some finite $F \subseteq E$. Show that a sequence (μ_n) of probability measures converges weakly if $\mu_n(A)$ converges for every increasing cylinder event A.

2.4. Let $G = (V, E)$ be a finite connected subgraph of the finite connected graph G'. Let T and T' be uniform spanning trees on G and G' respectively. Show that, for any edge e of G, $P(e \in T) \ge P(e \in T')$.

More generally, let B be a subset of E, and show that $P(B \subseteq T) \ge P(B \subseteq T')$.

2.5. Let T_n be a UST of the lattice box $[-n, n]^d$ of $\mathbb{Z}^d$. Show that the limit $\lambda(e) = \lim_{n\to\infty} P(e \in T_n)$ exists.

More generally, show that the weak limit of T_n exists as $n \to \infty$.

2.6. Adapt the conclusions of the last two examples to the 'wired' UST measure μ^{w} on $\mathbb{L}^d$.

2.7. Let $\mathcal{F}$ be the set of forests of $\mathbb{L}^d$ with no bounded component, and let μ be an automorphism-invariant probability measure with support $\mathcal{F}$. Show that the mean degree of every vertex is 2.

2.8. [170] Let A be an increasing cylinder event in $\{0, 1\}^{\mathbb{E}^d}$. Using the Feder–Mihail Theorem 2.5 or otherwise, show that the free and wired UST measures on $\mathbb{L}^d$ satisfy $\mu^{\mathrm{f}}(A) \ge \mu^{\mathrm{w}}(A)$. Deduce by the last exercise and Strassen's theorem, or otherwise, that $\mu^{\mathrm{f}} = \mu^{\mathrm{w}}$.

2.9. Consider the square lattice $\mathbb{L}^2$ as an infinite electrical network with unit edge-resistances. Show that the effective resistance between two neighbouring vertices is 2.

2.10. Let $G = (V, E)$ be finite and connected, and let $W \subseteq V$. Let $\mathcal{F}_W$ be the set of forests of G comprising exactly $|W|$ trees with respective roots the members of W. Explain how Wilson's algorithm may be adapted to sample uniformly from $\mathcal{F}_W$.

3

Percolation and Self-Avoiding Walk

The central feature of the percolation model is the phase transition. The existence of the point of transition is proved by path-counting and planar duality. Basic facts about self-avoiding walks, oriented percolation, and the coupling of models are reviewed.

3.1 Phase transition

Percolation is the fundamental stochastic model for spatial disorder. In its simplest form introduced in [47][1], it inhabits a (crystalline) lattice and possesses the maximum of (statistical) independence. We shall consider mostly percolation on the (hyper)cubic lattice $\mathbb{L}^d = (\mathbb{Z}^d, \mathbb{E}^d)$ in $d \geq 2$ dimensions, but much of the following may be adapted to an arbitrary lattice.

Percolation comes in two forms, 'bond' and 'site', and we concentrate here on the bond model. Let $p \in [0, 1]$. Each edge $e \in \mathbb{E}^d$ is designated either *open* with probability p, or *closed* otherwise, different edges receiving independent designations. We think of an open edge as being open to the passage of some material such as disease, liquid, or infection; closed edges are closed to such passage. Suppose we remove all closed edges, and consider the remaining open subgraph of the lattice. Percolation theory is concerned with the geometry of this open graph. Of particular interest are such quantites as the size of the open cluster C_x containing a given vertex x, and particularly the probability that C_x is infinite.

The sample space is the set $\Omega = \{0, 1\}^{\mathbb{E}^d}$ of 0/1-vectors ω indexed by the edge-set; here, 1 represents 'open', and 0 'closed'. As σ-field we take that generated by the finite-dimensional cylinder sets, and the relevant probability measure is product measure $\mathbb{P}_p$ with density p.

For $x, y \in \mathbb{Z}^d$, we write $x \leftrightarrow y$ if there exists an open path joining x and y. The *open cluster* C_x at x is the set of all vertices reachable along open paths from

[1] See also [214].

the vertex x:

$$C_x = \{y \in \mathbb{Z}^d : x \leftrightarrow y\}.$$

The origin of $\mathbb{Z}^d$ is denoted 0, and we write $C = C_0$. The principal object of study is the *percolation probability* $\theta(p)$ given by

$$\theta(p) = \mathbb{P}_p(|C| = \infty).$$

The critical probability is defined as

$$p_c = p_c(\mathbb{L}^d) = \sup\{p : \theta(p) = 0\}. \tag{3.1}$$

It is fairly clear (and we will spell this out soon) that θ is non-decreasing in p, and thus

$$\theta(p) \begin{cases} = 0 & \text{if } p < p_c, \\ > 0 & \text{if } p > p_c. \end{cases}$$

It is fundamental that $0 < p_c < 1$, and we state this as a theorem. It is easy to see that $p_c = 1$ for the corresponding one-dimensional process.

(3.2) Theorem. *For $d \geq 2$, we have that* $0 < p_c < 1$.

It is an important open problem to prove the following conjecture. The conclusion is known only for $d = 2$ and $d \geq 19$.

(3.3) Conjecture. *For $d \geq 2$, we have that* $\theta(p_c) = 0$.

It is the edges (or 'bonds') of the lattice that are declared open/closed above. If, instead, we designate the vertices (or 'sites') to be open/closed, the ensuing model is termed *site percolation*. Subject to minor changes, the theory of site percolation may be developed just as that of bond percolation.

Proof of Theorem 3.2. This proof introduces two basic methods, namely the counting of paths and the use of planar duality. We show first by counting paths that $p_c > 0$.

A *self-avoiding walk* (SAW) is a lattice path that visits no vertex more than once. Let σ_n be the number of SAWs with length n beginning at the origin, and let N_n be the number of such SAWs all of whose edges are open. Then

$$\begin{aligned} \theta(p) &= \mathbb{P}_p(N_n \geq 1 \text{ for all } n \geq 1) \\ &= \lim_{n\to\infty} \mathbb{P}_p(N_n \geq 1). \end{aligned}$$

Now,

$$\mathbb{P}_p(N_n \geq 1) \leq \mathbb{P}_p(N_n) = p^n \sigma_n. \tag{3.4}$$

A a crude upper bound for σ_n, we have that

$$\sigma_n \leq 2d(2d-1)^{n-1}, \qquad n \geq 1, \tag{3.5}$$

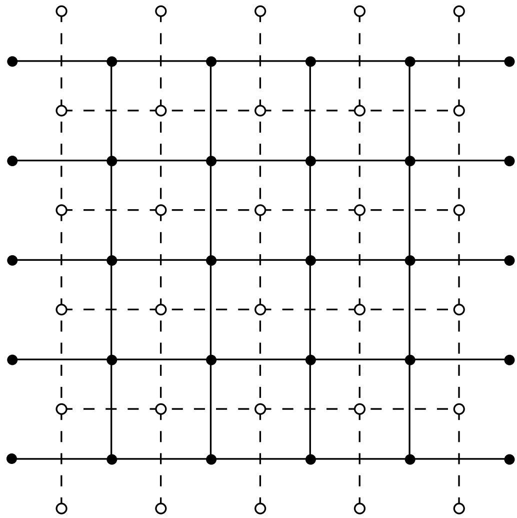

Figure 3.1. Part of the square lattice $\mathbb{L}^2$ and its dual.

since the first step of a SAW from the origin can be to any of its $2d$ neighbours, and there are no more than $2d-1$ choices for each subsequent step. Thus

$$\theta(p) \le \lim_{n\to\infty} 2d(2d-1)^{n-1}p^n,$$

which equals 0 whenever $p(2d-1) < 1$. Therefore,

$$p_c \ge \frac{1}{2d-1}.$$

We turn now to the proof that $p_c < 1$. The first step is to observe that

(3.6) $$p_c(\mathbb{L}^d) \ge p_c(\mathbb{L}^{d+1}), \qquad d \ge 2.$$

This follows immediately by the observation that $\mathbb{L}^d$ may be embedded in $\mathbb{L}^{d+1}$ in such a way that the origin lies in an infinite open cluster of $\mathbb{L}^{d+1}$ whenever it lies in an infinite open cluster of the smaller lattice $\mathbb{L}^d$. By (3.6), it suffices to show that

(3.7) $$p_c(\mathbb{L}^2) < 1,$$

and to this end we shall use planar duality. The square lattice has a special property, namely that of *self-duality*. Planar duality arises as follows. Let G be a planar graph, drawn in the plane. The *planar dual* of G is the graph constructed in the following way. We place a vertex in every face of G (including the infinite face if it exists) and we join two such vertices by an edge if and only if the corresponding faces of G share a boundary edge. It is easy to see that the dual of the square lattice

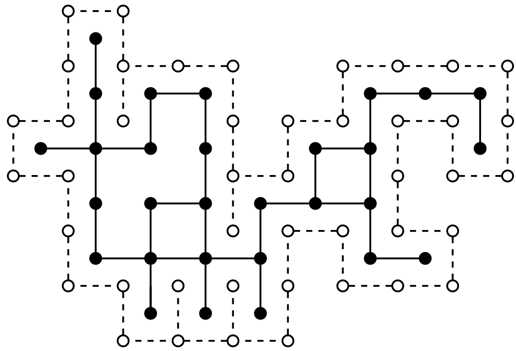

Figure 3.2. A finite open cluster of the primal lattice lies 'just inside' an closed cycle of the dual lattice.

$\mathbb{L}^2$ is a copy of $\mathbb{L}^2$, and we refer therefore to the square lattice as being *self-dual*. See Figure 3.1.

There is a natural one–one correspondence between the edge-set of the dual lattice $\mathbb{L}^2_{\mathrm{d}}$ and that of the primal $\mathbb{L}^2$, and this gives rise to a percolation model on $\mathbb{L}^2_{\mathrm{d}}$ by: for an edge $e \in \mathbb{E}^2$ and it dual edge e_{d}, we declare e_{d} to be open if and only if e is open. As illustrated in Figure 3.2, each finite open cluster of $\mathbb{L}^2$ lies in the interior of a closed cycle of $\mathbb{L}^2_{\mathrm{d}}$.

We use a 'Peierls argument' to obtain (3.7)[2]. Let M_n be the number of closed circuits of the dual lattice, having length n and containing 0 in their interior. Note that $|C| < \infty$ if and only if $M_n \geq 1$ for some n. Therefore,

$$
\begin{aligned}
(3.8) \qquad 1 - \theta(p) = \mathbb{P}_p(|C| < \infty) &= \mathbb{P}_p\Bigl(\sum_n M_n \geq 1\Bigr) \\
&\leq \mathbb{P}_p\Bigl(\sum_n M_n\Bigr) \\
&= \sum_{n=4}^{\infty} \mathbb{P}_p(M_n) \leq \sum_{n=4}^{\infty} (n4^n)(1-p)^n,
\end{aligned}
$$

where we have used the facts that the shortest dual circuit containing 0 has length 4, and that the total number of dual circuits, having length n and surrounding the origin, is no greater than $n4^n$. The final sum may be made strictly smaller than 1 by choosing p sufficiently close to 1, say $p > 1 - \epsilon$ where $\epsilon > 0$. This implies that $p_{\mathrm{c}}(\mathbb{L}^2) < 1 - \epsilon$ as required for (3.7). □

[2]This method was used by Peierls [169] to prove phase transition for the two-dimensional Ising model.

3.2 Self-avoiding walks

How many self-avoiding walks of length n exist, starting from the origin? What is the 'shape' of a SAW chosen at random from this set? In particular, what is the distance between its endpoints? These and related questions have attracted a great deal of attention since the publication in 1954 of the pioneering paper [116] of Hammersley and Morton, and never more so than in recent years. It is believed but not proved that a typical SAW on $\mathbb{L}^2$, starting at the origin, converges in a suitable manner as $n \to \infty$ to a SLE$_{8/3}$ curve, and the proof of this statement is an open problem of outstanding interest. See Section 2.5, in particular Figure 2.1, and [159, 190, 198].

The use of subadditivity was one of the several stimulating ideas of [116], and it has proved extremely fruitful in many contexts since. Consider the lattice $\mathbb{L}^d$, and let $\mathcal{S}_n$ be the set of SAWs with length n starting at the origin, and $\sigma_n = |\mathcal{S}_n|$ as before.

(3.9) Lemma. *We have that $\sigma_{m+n} \le \sigma_m \sigma_n$, for $m, n \ge 0$.*

Proof. Let π and π' be finite SAWs starting at the origin, and denote by $\pi * \pi'$ the walk obtained by following π from 0 to its other endpoint x, and then following the translated walk $\pi' + x$. Every $\nu \in \mathcal{S}_{m+n}$ may be written in a unique way as $\nu = \pi * \pi'$ for some $\pi \in \mathcal{S}_m$ and $\pi' \in \mathcal{S}_n$. The claim of the lemma follows. □

(3.10) Theorem [116]. *The limit $\kappa = \lim_{n\to\infty}(\sigma_n)^{1/n}$ exists and satisfies $d \le \kappa \le 2d - 1$.*

This is in essence a consequence of the 'sub-multiplicative' inequality of Lemma 3.9, see Exercise 3.1. The constant κ is called the *connective constant* of the lattice. The exact value of $\kappa = \kappa(\mathbb{L}^d)$ is unknown for every $d \ge 2$, see Section 7.2 of [126], pp. 481–483. It is conjectured that the connective constant of the 'hexagonal' (or 'honeycomb') lattice, illustrated in Figure 5.9, equals $\sqrt{2+\sqrt{2}}$.

Proof. By Lemma 3.9, $x_m = \log \sigma_m$ satisfies the 'subadditive inequality'

$$x_{m+n} \le x_m + x_n. \tag{3.11}$$

The existence of the limit

$$\lambda = \lim_{n\to\infty} \left\{ \frac{1}{n} x_n \right\}$$

follows immediately, and $\lambda = \inf_m\{x_m/m\} \in [-\infty, \infty)$. By (3.5), $\kappa = e^\lambda \le 2d - 1$. Finally, σ_n is at least the number of 'stiff' walks every step of which is in the direction of an *increasing* coordinate. The number of such walks is d^n, and therefore $\kappa \ge d$. □

The bounds of Theorem 3.2 may be improved as follows.

(3.12) Theorem. *The critical probability of bond percolation on* $\mathbb{L}^d$*, with* $d \ge 2$*, satisfies*

$$\frac{1}{\kappa(d)} \le p_c \le 1 - \frac{1}{\kappa(2)},$$

where $\kappa(d)$ *denotes the connective constant of* $\mathbb{L}^d$*.*

Proof. As in (3.4),

$$\theta(p) \le \lim_{n\to\infty} p^n \sigma_n.$$

Now, $\sigma_n = \kappa(d)^{(1+o(1))n}$, so that $\theta(p) = 0$ if $p\kappa(d) < 1$.

For the upper bound, we elaborate on the proof of the corresponding part of Theorem 3.2. Let F_m be the event that there exists a closed cycle of the dual lattice $\mathbb{L}^2_d$ containing the primal box $B(m) = [-m, m]^2$ in its interior, and let G_m be the event that all edges of $B(m)$ are open. These two events are independent, since they are defined in terms of disjoint sets of edges. As in (3.8),

$$\text{(3.13)} \qquad \mathbb{P}_p(F_m) \le \mathbb{P}_p\left(\sum_{n=4m}^{\infty} M_n \ge 1\right) \le \sum_{n=4m}^{\infty} n(1-p)^n \sigma_n.$$

Recall that $\sigma_n = \kappa(2)^{(1+o(1))n}$, and choose p such that $(1-p)\kappa(2) < 1$. By (3.13), we may find m such that $\mathbb{P}_p(F_m) < \frac{1}{2}$. Then,

$$\theta(p) \ge \mathbb{P}_p(\overline{F_m} \cap G_m) = \mathbb{P}_p(\overline{F_m})\mathbb{P}_p(G_m) \ge \tfrac{1}{2}\mathbb{P}_p(G_m) > 0.$$

The upper bound on p_c follows. □

There are some extraordinary conjectures concerning SAWs in two dimensions. We mention the conjecture that

$$\sigma_n \sim A n^{11/32} \kappa^n \qquad \text{when } d = 2,$$

expected to hold for any lattice in two dimensions, with an appropriate choice of constant A depending on the choice of lattice. It is known in contrast that no polynomial correction is necessary when $d \ge 5$,

$$\sigma_n \sim A\kappa^n \qquad \text{when } d \ge 5,$$

for the cubic lattice at least. See [159, 190, 198] for further details of these and other conjectures and results.

3.3 Coupled percolation

The use of coupling in probability theory goes back at least as far as the beautiful proof by Doeblin of the ergodic theorem for Markov chains, [65]. In percolation, we couple together the bond models with different values of p as follows. Let U_e, $e \in \mathbb{E}^d$, be independent random variables with the uniform distribution on [0, 1]. For $p \in [0, 1]$, let

$$\eta_p(e) = \begin{cases} 1 & \text{if } U_e < p, \\ 0 & \text{otherwise.} \end{cases}$$

Thus the configuration η_p ($\in \Omega$) has law $\mathbb{P}_p$, and in addition

$$\eta_p \le \eta_r \quad \text{if} \quad p \le r.$$

(3.14) Theorem. *For any increasing non-negative random variable $f : \Omega \to \Omega$, the function $g(p) = \mathbb{P}_p(f)$ is non-decreasing.*

Proof. For $p \le r$,

$$g(p) = \mathbb{P}(f(\eta_p)) \le \mathbb{P}(f(\eta_r)) = g(r),$$

as required, where $\mathbb{P}$ denotes 'generic probability'. □

3.4 Oriented percolation

The 'north–east' lattice $\vec{\mathbb{L}}^d$ obtained by orienting each edge of $\mathbb{L}^d$ in the direction of increasing coordinate-value (see Figure 3.3 for a two-dimensional illustration). There are many parallels between results for oriented percolation and those for ordinary percolation; on the other hand the corresponding proofs often differ, largely because the existence of one-way streets restricts the degree of spatial freedom of the traffic.

Let $p \in [0, 1]$. We declare an edge of $\vec{\mathbb{L}}^d$ to be *open* with probability p and otherwise *closed*. The states of different edges are taken to be independent. We now supply fluid at the origin, and allow it to travel along open edges in the directions of their orientations only. Let $\vec{C}$ be the set of vertices that may be reached from the origin along open directed paths. The *percolation probability* is

(3.15) $$\vec{\theta}(p) = \mathbb{P}_p(|\vec{C}| = \infty),$$

and the critical probability $\vec{p}_c(d)$ by

(3.16) $$\vec{p}_c(d) = \sup\{p : \vec{\theta}(p) = 0\}.$$

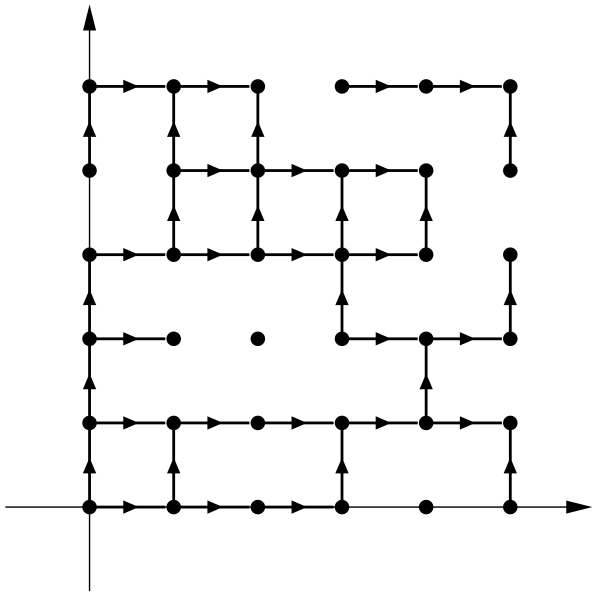

Figure 3.3. Part of the two-dimensional 'north–east' lattice in which each edge has been deleted with probability $1 - p$, independently of all other edges.

(3.17) Theorem. *For $d \geq 2$, we have that* $0 < \vec{p}_c(d) < 1$.

Proof. Since an oriented path is also a path, it is immediate that $\vec{\theta}(p) \leq \theta(p)$, whence $\vec{p}_c(d) \geq p_c$. As in the proof of Theorem 3.2, it suffices for the converse to show that $\vec{p}_c = \vec{p}_c(2) < 1$.

Let $d = 2$. The cluster $\vec{C}$ comprises the endvertices of open edges that are oriented northwards/eastwards. Assume $|\vec{C}| < \infty$. Surrounding $\vec{C}$ one may draw a cycle Δ of the dual in the manner illustrated in Figure 3.4. As we traverse Δ in the clockwise direction, we traverse edges each of which is oriented in one of the four compass directions. Any edge of Δ that is oriented either eastwards or southwards crosses a primal edge that is closed. Exactly one half of the edges of Δ are oriented thus, so that, as in (3.8),

$$\mathbb{P}_p(|\vec{C}| < \infty) \leq \sum_{n \geq 4} 4 \cdot 3^{n-2}(1-p)^{\frac{1}{2}n-1}.$$

In particular, $\vec{\theta}(p) > 0$ if $1 - p$ is sufficiently small and positive. □

The process is understood quite well when $d = 2$, see [70]. By looking at the set A_n of wet vertices on the diagonal $\{x \in \mathbb{Z}^2 : x_1 + x_2 = n\}$ of $\vec{\mathbb{L}}^2$, one may reformulate two-dimensional oriented percolation as a one-dimensional contact process in discrete time (see [148], Chapter 6)). It turns out that $\vec{p}_c(2)$ may be characterized in terms of the velocity of the rightwards edge of a contact process on $\mathbb{Z}$ whose initial distribution places infectives to the left of the origin and

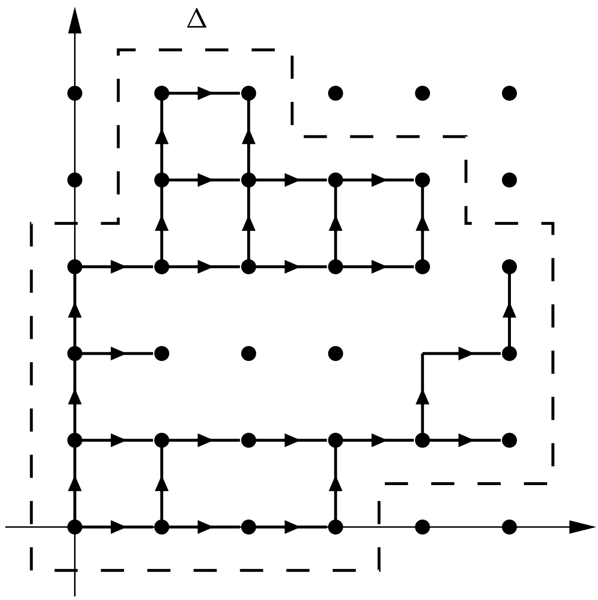

Figure 3.4. As we trace the dual cycle Δ, we traverse edges exactly one half of which cross closed boundary edges of the cluster $\vec{C}$ at the origin.

susceptibles to the right. With the support of arguments from branching processes and ordinary percolation, one may prove such results as the exponential decay of the cluster size distribution when $p < \vec{p}_c(2)$, and its sub-exponential decay when $p > \vec{p}_c(2)$: there exist $\alpha(p), \beta(p) > 0$ such that

$$e^{-\alpha(p)\sqrt{n}} \le \mathbb{P}_p\big(n \le |\vec{C}| < \infty\big) \le e^{-\beta(p)\sqrt{n}} \qquad \text{if } \vec{p}_c(2) < p < 1. \tag{3.18}$$

There is a close relationship between oriented percolation and the contact model (see Chapter 6), and methods developed for the latter model may often be applied to the former. It has been shown in particular that $\vec{\theta}(\vec{p}_c) = 0$ for general $d \ge 2$, see [100].

We close this section with an open problem of a different sort. Suppose that each edge of $\mathbb{L}^2$ is oriented in a random direction, horizontal edges being oriented eastwards with probability p and westwards otherwise, and vertical edges being oriented northwards with probability p and southwards otherwise. Let $\eta(p)$ be the probability that there exists an infinite oriented path starting at the origin. It is not hard to show that $\eta(\frac{1}{2}) = 0$ (see Exercise 3.9. We ask whether or not $\eta(p) > 0$ if $p \ne \frac{1}{2}$. Partial results in this direction may be found in [97].

3.5 Exercises

3.1. *Subadditive inequality.* Let $(x_n : n \geq 1)$ be a real sequence satisfying $x_{m+n} \leq x_m + x_n$ for $m, n \geq 1$. Show that the limit $\lambda = \lim_{n\to\infty}\{x_n/n\}$ exists and satisfies $\lambda = \inf_k\{x_k/k\}$.

3.2. (continuation) Find reasonable conditions on the sequence (α_n) such that: the generalized inequality

$$x_{m+n} \leq x_m + x_n + \alpha_m, \qquad m, n \geq 1,$$

implies the existence of the limit $\lambda = \lim_{n\to\infty}\{x_n/n\}$?

3.3. [108] *Bond/site critical probabilities.* Let G be an infinite connected graph with maximal vertex degree Δ. Show that the critical probabilities for bond and site percolation on G satisfy

$$p_c^{\text{bond}} \leq p_c^{\text{site}} \leq 1 - [1 - p_c^{\text{bond}}]^{\Delta}.$$

The second inequality is in fact valid with Δ replaced by $\Delta - 1$.

3.4. Show that bond percolation on a graph G may be reformulated in terms of site percolation on a graph derived suitably from G.

3.5. Show that the connective constant of $\mathbb{L}^2$ lies strictly between 2 and 3.

3.6. *One-dimensional percolation.* Each edge of the one-dimensional lattice $\mathbb{Z}$ is declared *open* with probability p. For $k \in \mathbb{Z}$, let $r(k) = \max\{u : k \leftrightarrow k+u\}$, and $L_n = \max\{r(k) : 1 \leq k \leq n\}$. Show that $\mathbb{P}_p(L_n > u) \leq np^u$, and deduce that, for $\epsilon > 0$,

$$\mathbb{P}_p\left(L_n > \frac{(1+\epsilon)\log n}{\log(1/p)}\right) \to 0 \qquad \text{as } n \to \infty.$$

[This is the famous problem of the longest run of heads in n tosses of a coin.]

3.7. (continuation) Show that, for $\epsilon > 0$,

$$\mathbb{P}_p\left(L_n < \frac{(1-\epsilon)\log n}{\log(1/p)}\right) \to 0 \qquad \text{as } n \to \infty.$$

By suitable refinements of the error estimates above, show that

$$\mathbb{P}_p\left(\frac{(1-\epsilon)\log n}{\log(1/p)} < L_n < \frac{(1+\epsilon)\log n}{\log(1/p)}, \text{ for all but finitely many } n\right) = 1.$$

3.8. Show the strict inequality $p_c(d) < \vec{p}_c(d)$ for the critical probabilities of unoriented and oriented percolation on $\mathbb{L}^d$ with $d \geq 2$.

3.9. [97] Each edge of the square lattice $\mathbb{L}^2$ is oriented in a random direction, horizontal edges being oriented eastwards with probability p and westwards otherwise, and vertical edges being oriented northwards with probability p and southwards otherwise. Let $\eta(p)$ be the probability that there exists an infinite oriented path starting at the origin. Show that $\eta(\frac{1}{2}) = 0$.

3.10. The vertex (i, j) of $\mathbb{L}^2$ is called *even* if $i + j$ is even, and *odd* otherwise. Vertical edges are oriented from the even endpoint to the odd, and horizontal edges vice versa. Each edge is declared *open* with probability p, and closed otherwise (independently between edges). Show that, for p sufficiently close to 1, there is strictly positive probability that the origin is the endpoint of an infinite open oriented path.

3.11. A *word* is an element of the set $\{0, 1\}^{\mathbb{N}}$ of singly-infinite $0/1$ sequences. Let $p \in (0, 1)$ and $M \geq 1$. Consider oriented site percolation on $\mathbb{Z}^2$, in which the colour $\omega(x)$ of a vertex x equals 1 with probability p, and 0 otherwise. A word $w = (w_1, w_2, \dots)$ is said to be *M-seen* if there exists an infinite oriented path $x_0 = 0, x_1, x_2, \dots$ of vertices such that $\omega(x_i) = w_i$ and $d(x_{i-1}, x_i) \leq M$ for $i \geq 1$. [Here, as usual, d denotes graph-theoretic distance.]

Calculate the probability that the square $\{1, 2, \dots, k\}^2$ contains both a 0 and a 1. Deduce[3] by a block argument that

$$\psi_p(M) = P_p(\text{all words are } M\text{-seen})$$

satisfies $\psi_p(M) > 0$ for $M \geq M(p)$, and determine an upper bound on the required $M(p)$.

[3] This provides a short proof of the main result of [152].

4

Correlation and Concentration

Correlation-type inequalities have played a significant role in the theory of disordered spatial systems. The Holley inequality provides a sufficient condition for the stochastic ordering of two measures, and also a route to a proof of the famous FKG inequality. For product measures, the complementary BK inequality involves the concept of 'disjoint occurrence'. Two concepts of concentration are considered here. The Hoeffding inequality provides a bound on the tail of a martingale with bounded differences. Another concept of 'influence' proved by Kahn, Kalai, and Linial leads to sharp-threshold theorems for increasing events under either product or FKG measures.

4.1 Holley inequality

We review the stochastic ordering of probability measures on a discrete space. Let E be a non-empty finite set, and $\Omega = \{0, 1\}^E$. The sample space Ω is partially ordered by:

$$\omega_1 \le \omega_2 \quad \text{if} \quad \omega_1(e) \le \omega_2(e) \text{ for all } e \in E.$$

A non-empty subset $A \subseteq \Omega$ is called *increasing* if:

$$\omega \in A, \ \omega \le \omega' \quad \Rightarrow \quad \omega' \in A.$$

The subset A is *decreasing* if its complement $\overline{A} = \Omega \setminus A$ is increasing.

(4.1) Definition. *Given two probability measures μ_i, $i = 1, 2$, on Ω, we write $\mu_1 \le_{\text{st}} \mu_2$ if*

$$\mu_1(A) \le \mu_2(A) \quad \textit{for all increasing events } A.$$

Equivalently, $\mu_1 \le_{\text{st}} \mu_2$ if and only if $\mu_1(f) \le \mu_2(f)$ for all increasing functions $f : \Omega \to \mathbb{R}$. There is an important and useful result, often termed Strassen's theorem, that asserts that measures satisfying $\mu_1 \le_{\text{st}} \mu_2$ may be coupled in a 'pointwise monotone' manner. Such a statement is valid for very general spaces (see [153]), but we restrict ourselves here to the current context. The proof is omitted, and may be found in many places including [157].

(4.2) Theorem [201]. *Let μ_1 and μ_2 be probability measures on Ω. The following two statements are equivalent.*

(i) $\mu_1 \le_{\mathrm{st}} \mu_2$.

(ii) *There exists a probability measure ν on Ω^2 such that $\nu(\{(\pi, \omega) : \pi \le \omega\}) = 1$, and whose marginal measures are μ_1 and μ_2.*

For $\omega_1, \omega_2 \in \Omega$, we define the (pointwise) maximum and minimum configurations by

$$\begin{aligned} \omega_1 \vee \omega_2(e) &= \max\{\omega_1(e), \omega_2(e)\}, \\ \omega_1 \wedge \omega_2(e) &= \min\{\omega_1(e), \omega_2(e)\}, \end{aligned} \tag{4.3}$$

for $e \in E$. A probability measure μ on Ω is called *positive* if $\mu(\omega) > 0$ for all $\omega \in \Omega$.

(4.4) Holley inequality [125]. *Let μ_1 and μ_2 be positive probability measures on Ω satisfying*

$$\mu_2(\omega_1 \vee \omega_2)\mu_1(\omega_1 \wedge \omega_2) \ge \mu_1(\omega_1)\mu_2(\omega_2), \qquad \omega_1, \omega_2 \in \Omega. \tag{4.5}$$

Then $\mu_1 \le_{\mathrm{st}} \mu_2$.

Proof. The main step is the proof that μ_1 and μ_2 can be 'coupled' in such a way that the component with marginal measure μ_2 lies above (in the sense of sample realizations) that with marginal measure μ_1. This is achieved by constructing a certain Markov chain with the coupled measure as unique invariant measure.

Here is a preliminary calculation. Let μ be a positive probability measure on Ω. We can construct a time-reversible Markov chain with state space Ω and unique invariant measure μ by choosing a suitable generator G satisfying the detailed balance equations. The dynamics of the chain involve the 'switching on or off' of components of the current state.

For $\omega \in \Omega$ and $e \in E$, we define the configurations ω^e, ω_e by

$$\omega^e(f) = \begin{cases} \omega(f) & \text{if } f \ne e, \\ 1 & \text{if } f = e, \end{cases} \qquad \omega_e(f) = \begin{cases} \omega(f) & \text{if } f \ne e, \\ 0 & \text{if } f = e. \end{cases} \tag{4.6}$$

Let $G : \Omega_E^2 \to \mathbb{R}$ be given by

$$G(\omega_e, \omega^e) = 1, \quad G(\omega^e, \omega_e) = \frac{\mu(\omega_e)}{\mu(\omega^e)}, \tag{4.7}$$

for all $\omega \in \Omega$, $e \in E$. Set $G(\omega, \omega') = 0$ for all other pairs ω, ω' with $\omega \ne \omega'$. The diagonal elements are chosen in such a way that

$$\sum_{\omega' \in \Omega} G(\omega, \omega') = 0, \qquad \omega \in \Omega.$$

It is elementary that

$$\mu(\omega)G(\omega,\omega') = \mu(\omega')G(\omega',\omega), \qquad \omega,\omega' \in \Omega,$$

and therefore G generates a time-reversible Markov chain on the state space Ω. This chain is irreducible (using (4.7)), and therefore possesses a unique invariant measure μ (see [109], Theorem 6.5.4).

We next follow a similar route for *pairs* of configurations. Let μ_1 and μ_2 satisfy the hypotheses of the theorem, and let S be the set of all pairs (π,ω) of configurations in Ω satisfying $\pi \le \omega$. We define $H : S \times S \to \mathbb{R}$ by

$$H(\pi_e,\omega;\pi^e,\omega^e) = 1, \tag{4.8}$$

$$H(\pi,\omega^e;\pi_e,\omega_e) = \frac{\mu_2(\omega_e)}{\mu_2(\omega^e)}, \tag{4.9}$$

$$H(\pi^e,\omega^e;\pi_e,\omega^e) = \frac{\mu_1(\pi_e)}{\mu_1(\pi^e)} - \frac{\mu_2(\omega_e)}{\mu_2(\omega^e)}, \tag{4.10}$$

for all $(\pi,\omega) \in S$ and $e \in E$; all other off-diagonal values of H are set to 0. The diagonal terms are chosen in such a way that

$$\sum_{\pi',\omega'} H(\pi,\omega;\pi',\omega') = 0, \qquad (\pi,\omega) \in S.$$

Equation (4.8) specifies that, for $\pi \in \Omega$ and $e \in E$, the edge e is acquired by π (if it does not already contain it) at rate 1; any edge so acquired is added also to ω if it does not already contain it. (Here, we speak of a configuration ψ containing an edge e if $\psi(e) = 1$.) Equation (4.9) specifies that, for $\omega \in \Omega$ and $e \in E$ with $\omega(e) = 1$, the edge e is removed from ω (and also from π if $\pi(e) = 1$) at the rate given in (4.9). For e with $\pi(e) = 1$, there is an additional rate given in (4.10) at which e is removed from π but not from ω. We need to check that this additional rate is indeed non-negative, and the required inequality,

$$\mu_2(\omega^e)\mu_1(\pi_e) \ge \mu_1(\pi^e)\mu_2(\omega_e), \qquad \pi \le \omega$$

follows from assumption (4.5).

Let $(X_t,Y_t)_{t\ge 0}$ be a Markov chain on S with generator H, and set $(X_0,Y_0) = (0,1)$, where 0 (respectively, 1) is the state of all 0's (respectively, 1's). By examination of (4.8)–(4.10) we see that $X = (X_t)_{t\ge 0}$ is a Markov chain with generator given by (4.7) with $\mu = \mu_1$, and that $Y = (Y_t)_{t\ge 0}$ arises similarly with $\mu = \mu_2$.

Let κ be an invariant measure for the paired chain $(X_t,Y_t)_{t\ge 0}$. Since X and Y have (respective) unique invariant measures μ_1 and μ_2, the marginals of κ are μ_1 and μ_2. We have by construction that $\kappa(S) = 1$, and κ is the required 'coupling' of μ_1 and μ_2.

Let $(\pi,\omega) \in S$ be chosen according to the measure κ. Then

$$\mu_1(f) = \kappa(f(\omega)) \le \kappa(f(\pi)) = \mu_2(f),$$

for any increasing function f. Therefore $\mu_1 \le_{\text{st}} \mu_2$. □

4.2 FKG inequality

The FKG inequality for product measures was discovered by Harris [121], and is often named now after the authors of [83] who proved the more general version that is the subject of this section. See the appendix of [98] for a historical account. Let E be a finite set, and $\Omega = \{0, 1\}^E$ as usual.

(4.11) Theorem. FKG inequality [83]. *Let μ be a positive probability measure on Ω such that*

$$\mu(\omega_1 \vee \omega_2)\mu(\omega_1 \wedge \omega_2) \ge \mu(\omega_1)\mu(\omega_2), \qquad \omega_1, \omega_2 \in \Omega. \tag{4.12}$$

Then μ is 'positively associated' in that

$$\mu(fg) \ge \mu(f)\mu(g) \tag{4.13}$$

for all increasing random variables $f, g : \Omega \to \mathbb{R}$.

It is explained in [83] how the condition of (strict) positivity can be removed. Condition (4.12) is called the 'FKG lattice condition'.

Proof. Assume that μ satisfies (4.12), and let f and g be increasing functions. By adding a constant to the function g, we see that it suffices to prove (4.13) under the additional hypothesis that g is strictly positive. We assume this holds. Define positive probability measures μ_1 and μ_2 on Ω by $\mu_1 = \mu$ and

$$\mu_2(\omega) = \frac{g(\omega)\mu(\omega)}{\sum_{\omega'} g(\omega')\mu(\omega')}, \qquad \omega \in \Omega.$$

Since g is increasing, (4.5) follows from (4.12). By the Holley inequality, Theorem 4.4,

$$\mu_1(f) \le \mu_2(f),$$

which is to say that

$$\frac{\sum_{\omega} f(\omega)g(\omega)\mu(\omega)}{\sum_{\omega'} g(\omega')\mu(\omega')} \ge \sum_{\omega} f(\omega)\mu(\omega)$$

as required. □

4.3 BK inequality

In the special case of product measure on Ω, there is a type of converse inequality to the FKG inequality, named the BK inequality after the authors of [32]. This is based on a concept of 'disjoint occurrence' that we make more precise as follows.

For $\omega \in \Omega$ and $F \subseteq E$ we define the cylinder event $C(\omega, F)$ generated by ω on F by

$$\begin{aligned} C(\omega, F) &= \{\omega' \in \Omega : \omega'(e) = \omega(e) \text{ for all } e \in F\} \\ &= (\omega(e) : e \in F) \times \{0, 1\}^{E \setminus F}. \end{aligned}$$

We define the event $A \square B$ as the set of all $\omega \in \Omega$ for which there exists a set $F \subseteq E$ such that $C(\omega, F) \subseteq A$ and $C(\omega, E \setminus F) \subseteq B$. Thus, $A \square B$ is the set of configurations ω for which there exist disjoint sets F, G of indices with the property that: knowledge of ω restricted to F (respectively, G) implies that $\omega \in A$ (respectively, $\omega \in B$). In the special case when A and B are increasing, $C(\omega, F) \subseteq A$ if and only if $\omega_F \in A$, where

$$\omega_F(e) = \begin{cases} \omega(e) & \text{for } e \in F, \\ 0 & \text{for } e \notin F. \end{cases}$$

Thus, in this case, $A \square B = A \circ B$ where

$$A \circ B = \big\{\omega : \text{there exists } F \subseteq E \text{ such that } \omega_F \in A,\ \omega_{E \setminus F} \in B\big\}.$$

The set F is permitted to depend on the choice of configuration ω.

Note that $A \square B \subseteq A \cap B$. Furthermore, if A and B are increasing, then so is $A \square B$ $(= A \circ B)$.

Let P be the product measure on Ω with local densities p_e, $e \in E$, that is

$$P = \prod_{e \in E} \mu_e,$$

where $\mu_e(0) = 1 - p_e$ and $\mu_e(1) = p_e$.

(4.14) Theorem. BK inequality [32]. *For increasing subsets A, B of Ω,*

$$P(A \circ B) \le P(A)P(B). \tag{4.15}$$

It is not known for what non-product measures (4.15) holds. It seems reasonable, for example, to conjecture that (4.15) holds for the measure P_k that selects a k-subset of E uniformly at random. It would be very useful to show that the random-cluster measure $\phi_{p,q}$ on Ω satisfies (4.15) whenever $0 < q < 1$, although we may have to survive with rather less. See Chapter 8, and Section 3.9 of [98].

The conclusion of the BK inequality is in fact valid for all pairs A, B of events, regardless of whether or not they are increasing. This is much harder to prove, and has not yet been as valuable as originally expected in the analysis of disordered systems.

(4.16) Theorem. Reimer inequality [181]. *For* $A, B \subseteq \Omega$,

$$P(A \,\square\, B) \le P(A)P(B).$$

One can see that $A \,\square\, B = A \cap B$ if A is increasing and B is decreasing. By applying Reimer's inequality to the events A and $\overline{B}$, where A and B are increasing, we obtain that $P(A \cap B) \ge P(A)P(B)$. Therefore, Reimer's inequality includes both the FKG and BK inequalities for the product measure P.

Proof of Theorem 4.14. We present the 'simple' proof of [30], see also [95, 210]. Those who prefer proofs by induction are directed to [44]. Let $1, 2, \dots, N$ be an ordering of E. We shall consider the duplicated sample space $\Omega \times \Omega'$ where $\Omega = \Omega' = \{0, 1\}^E$, with which we associate the product measure $\overline{P} = P \times P$. Elements of Ω (respectively, Ω') are written as ω (respectively, ω'). Let A and B be increasing subsets of $\{0, 1\}^E$. For $j \ge 1$ and $(\omega, \omega') \in \Omega \times \Omega'$, define the N-vector ω_j by

$$\omega_j = \big(\omega'(1), \omega'(2), \dots, \omega'(j-1), \omega(j), \dots, \omega(N)\big),$$

and the events $\widehat{A}_j$, $\widehat{B}$ of $\Omega \times \Omega'$ by

$$\widehat{A}_j = \{(\omega, \omega') : \omega_j \in A\}, \qquad \widehat{B} = \{(\omega, \omega') : \omega \in B\}.$$

Note that:

(a) $\widehat{A}_1 = A \times \Omega'$ and $\widehat{B} = B \times \Omega'$, so that $\widehat{P}(\widehat{A}_1 \circ \widehat{B}) = P(A \circ B)$,

(b) $\widehat{A}_{N+1}$ and $\widehat{B}$ are defined in terms of disjoint subsets of E, so that

$$\widehat{P}(\widehat{A}_{N+1} \circ \widehat{B}) = \widehat{P}(\widehat{A}_{N+1})\widehat{P}(\widehat{B}) = P(A)P(B).$$

It thus suffices to show that

$$\widehat{P}(\widehat{A}_j \circ \widehat{B}) \le \widehat{P}(\widehat{A}_{j+1} \circ \widehat{B}), \qquad 1 \le j \le N, \tag{4.17}$$

and this we do, for given j, by conditioning on the values of the $\omega(i), \omega'(i)$ for all $i \ne j$. Suppose these values are given, and classify them as follows. There are three cases.

1. $\widehat{A}_j \circ \widehat{B}$ does not occur when $\omega(j) = \omega'(j) = 1$.
2. $\widehat{A}_j \circ \widehat{B}$ occurs when $\omega(j) = \omega'(j) = 0$, in which case $\widehat{A}_{j+1} \circ \widehat{B}$ occurs also.
3. Neither of the two cases above hold.

Consider the third case. Since $\widehat{A}_j \circ \widehat{B}$ does not depend on the value $\omega'(j)$, we have in this case that $\widehat{A}_j \circ B$ occurs if and only if $\omega(j) = 1$, and therefore the conditional probability of $\widehat{A}_j \circ \widehat{B}$ is p_j. When $\omega(j) = 1$, edge j is 'contributing' to either $\widehat{A}_j$ or $\widehat{B}$ but not both. Replacing $\omega(j)$ by $\omega'(j)$, we find similarly that the conditional probability of $\widehat{A}_{j+1} \circ \widehat{B}$ is at least p_j.

In each of the three cases above, the conditional probability of $\widehat{A}_j \circ \widehat{B}$ is no greater than that of $\widehat{A}_{j+1} \circ \widehat{B}$, and (4.17) follows. □

4.4 Hoeffding inequality

Let $(Y_n, \mathcal{F}_n)$, $n \ge 0$, be a martingale. One can obtain bounds for the tail of Y_n in terms of the sizes of the martingale differences $D_k = Y_k - Y_{k-1}$. These bounds are surprisingly tight, and they have had substantial impact in various areas of application, especially those with a combinatorial structure. We describe such a bound in this section for the case when the D_k are bounded random variables.

(4.18) Theorem. Hoeffding inequality. *Let $(Y_n, \mathcal{F}_n)$, $n \ge 0$, be a martingale such that $|Y_k - Y_{k-1}| \le K_k$ (a.s.) for all k and some real sequence (K_k). Then*

$$P(Y_n - Y_0 \ge x) \le \exp\left(-\tfrac{1}{2}x^2/L_n\right), \qquad x > 0.$$

where $L_n = \sum_{k=1}^n K_k^2$.

Since Y_n is a martingale, so is $-Y_n$, and thus the same bound is valid for $P(Y_n - Y_0 \le -x)$. Such inequalities are often named after Azuma [21] and Hoeffding [124]. See [161] for a review of the so-called 'method of bounded differences', and [109, Sect. 12.2], for some applications.

Theorem 4.18 is one of a family of inequalities much used in probabilistic combinatorics, in what is termed the 'method of bounded differences'. See the discussion in [161]. Its applications are of the following general form. Suppose that we are given N random variables $X_1, X_2, \dots, X_N$, and we wish to study the behaviour of some function $Z = Z(X_1, X_2, \dots, X_N)$. For example, the X_i might be the sizes of objects to be packed in bins, and Z the minimum number of bins required to pack them. Let $\mathcal{F}_n = \sigma(X_1, X_2, \dots, X_n)$, and define the martingale $Y_n = E(Z \mid \mathcal{F}_n)$. Thus, $Y_0 = E(Z)$ and $Y_N = Z$. If the martingale differences are bounded, Theorem 4.18 provides a bound for the tail probability $P(|Z - E(Z)| \ge x)$. We shall see an application of this type at Theorem 11.13, which deals with the chromatic number of random graphs.

Proof. The function $g(d) = e^{\psi d}$ is convex for $\psi > 0$, and therefore

(4.19) $$e^{\psi d} \le \tfrac{1}{2}(1-d)e^{-\psi} + \tfrac{1}{2}(1+d)e^{\psi} \qquad |d| \le 1.$$

Applying this to a random variable D having mean 0 and satisfying $P(|D| \le 1) = 1$, we obtain

(4.20) $$E(e^{\psi D}) \le \tfrac{1}{2}(e^{-\psi} + e^{\psi}) < e^{\frac{1}{2}\psi^2}, \qquad \psi > 0.$$

where the final inequality follows by a comparison of the coefficients of the ψ^{2n}.

By Markov's inequality,

(4.21) $$P(Y_n - Y_0 \ge x) \le e^{-\theta x} E(e^{\theta(Y_n - Y_0)}), \qquad \theta > 0.$$

With $D_n = Y_n - Y_{n-1}$,

$$E(e^{\theta(Y_n - Y_0)}) = E(e^{\theta(Y_{n-1} - Y_0)} e^{\theta D_n}).$$

Since $Y_{n-1} - Y_0$ is $\mathcal{F}_{n-1}$-measurable,

$$\begin{aligned} E(e^{\theta(Y_n - Y_0)} \mid \mathcal{F}_{n-1}) &= e^{\theta(Y_{n-1}-Y_0)} E(e^{\theta D_n} \mid \mathcal{F}_{n-1}) \\ &\le e^{\theta(Y_{n-1}-Y_0)} \exp(\tfrac{1}{2}\theta^2 K_n^2), \end{aligned} \tag{4.22}$$

where we have applied (4.20) to the random variable D_n/K_n at the last step. We take expectations of (4.22) and iterate to obtain

$$E(e^{\theta(Y_n-Y_0)}) \le E(e^{\theta(Y_{n-1}-Y_0)}) \exp(\tfrac{1}{2}\theta^2 K_n^2) \le \exp(\tfrac{1}{2}\theta^2 L_n).$$

Therefore, by (4.21),

$$P(Y_n - Y_0 \ge x) \le \exp(-\theta x + \tfrac{1}{2}\theta^2 L_n), \qquad \theta > 0.$$

Let $x > 0$, and set $\theta = x/L_n$ (this is the value that minimizes the exponent). Then

$$P(Y_n - Y_0 \ge x) \le \exp(-\tfrac{1}{2}x^2/L_n), \qquad x > 0,$$

as required. □

4.5 Influence for product measures

Let $N \ge 1$ and $E = \{1, 2, \dots, N\}$, and write $\Omega = \{0, 1\}^E$. Let μ be a probability measure on Ω, and A an event (that is, a subset of Ω). Two ways of defining the 'influence' of a member $e \in E$ on the event A come to mind. The (*conditional*) *influence* is defined to be

$$J_A(e) = \mu(A \mid \omega(e) = 1) - \mu(A \mid \omega(e) = 0). \tag{4.23}$$

The *absolute influence* is

$$I_A(e) = \mu(1_A(\omega^e) \ne 1_A(\omega_e)), \tag{4.24}$$

where 1_A is the indicator function of A, and ω^e, ω_e are the configurations given by (4.3). In a voting analogy, each of N voters has 1 vote, and A is the set of vote-vectors that result in a given outcome. Then $I_A(e)$ is the probability that voter e can influence the outcome.

We make two remarks concerning the above definitions. First, if A is increasing,

$$I_A(e) = \mu(A^e) - \mu(A_e), \tag{4.25}$$

where

$$A^e = \{\omega \in \Omega : \omega^e \in A\}, \quad A_e = \{\omega \in \Omega : \omega_e \in A\}.$$

If, in addition, μ is a product measure, then $I_A(e) = J_A(e)$. Note that influences depend on the underlying measure.

Let ϕ_p be product measure with density p on Ω, and write $\phi = \phi_{\frac{1}{2}}$, the uniform measure. All logarithms are taken to base 2 until further notice.

There has been extensive study of the largest (absolute) influence, $\max_e I_A(e)$, when μ is a product measure, and this has been used to obtain 'sharp threshold' theorems for the probability $\phi_p(A)$ of an increasing event A viewed as a function of p.

(4.26) Theorem (Influence) [130]. *There exists a constant $c \in (0, \infty)$ such that the following holds. Let $N \geq 1$, let E be a finite set with $|E| = N$, and let A be a subset of $\Omega = \{0, 1\}^E$ with $\phi(A) \in (0, 1)$. Then*

$$\sum_{e \in E} I_A(e) \geq c\phi(A)(1 - \phi(A)) \log[1/\max_e I_A(e)], \tag{4.27}$$

where the reference measure is $\phi = \phi_{\frac{1}{2}}$. There exists $e \in E$ such that

$$I_A(e) \geq c\phi(A)(1 - \phi(A)) \frac{\log N}{N}. \tag{4.28}$$

Note that

$$\phi(A)(1 - \phi(A)) \geq \tfrac{1}{2} \min\{\phi(A), 1 - \phi(A)\}.$$

We indicate at this stage the reason why (4.27) implies (4.28). We may assume that $m = \max_e I_A(e)$ satisfies $m > 0$, since otherwise $\phi(A)(1 - \phi(A)) = 0$. Since

$$\sum_{e \in E} I_A(e) \leq Nm,$$

we have by (4.27) that

$$\frac{m}{\log(1/m)} \geq \frac{c\phi(A)(1 - \phi(A))}{N}.$$

Inequality (4.28) follows with an amended value of c, by the monotonicity of $m/\log(1/m)$ or otherwise[1].

Such results have applications to several topics including random graphs, random walks, and percolation, see [132]. We summarize two such applications next, and we defer until Section 5.8 a complete application to site percolation on the triangular lattice.

I. *First-passage percolation* is concerned with passage times on a graph whose edges have random 'travel-times'. Suppose we assign to each edge e of the d-dimensional cubic lattice $\mathbb{L}^d$ a random travel-time T_e, the T_e being non-negative and independent with a common distribution function F. The passage time of a path π is the sum of the travel-times of its edges. Given two vertices u, v, the passage time $T_{u,v}$ is defined as the infimum of the passage times of the set of paths joining u to v. The main question is to understand the asymptotic properties of $T_{0,v}$ as $|v| \to \infty$. This model for the time-dependent flow of material was introduced in [117], and has been studied extensively since.

It is a consequence of the subadditive ergodic theorem that, subject to a suitable moment condition, the (deterministic) limit

$$\mu_v = \lim_{n \to \infty} \frac{1}{n} T_{0,nv}$$

[1]When $N = 1$, there is nothing to prove. This is left as an exercise when $N \geq 2$.

exists almost surely. Indeed, the subadditive ergodic theorem was conceived explicitly in order to prove such a statement for first-passage percolation. The constant μ_v is called the *time constant* in direction v. One of the open problems is to understand the asymptotic behaviour of $\mathrm{var}(T_{0,v})$ as $|v| \to \infty$. Various relevant results are known, and one of the best uses an influence theorem due to Talagrand [204], and related to Theorem 4.26. Specifically, it is proved in [28] that $\mathrm{var}(T_{0,v}) \le C|v|/\log|v|$ for some constant $C = C(a, b, d)$, in the situation when each T_e is equally likely to take either of the two positive values a, b. It has been predicted that $\mathrm{var}(T_{0,v}) \sim |v|^{2/3}$ when $d = 2$. This work has been continued in [26].

II. The *Voronoi percolation model* is a continuum model that we construct as follows in $\mathbb{R}^2$. Let Π be a Poisson process of intensity 1 in $\mathbb{R}^2$. With any $u \in \Pi$, we associate the 'tile'

$$T_u = \{x \in \mathbb{R}^2 : |x - u| \le |x - v| \text{ for all } v \in \Pi\}.$$

Two points $u, v \in \Pi$ are declared *adjacent*, written $u \sim v$, if T_u and T_v share a boundary segment. We now consider site percolation on the graph Π with this adjacency relation. It was long believed that the critical percolation probability of this model is $\frac{1}{2}$ (almost surely, with respect to the Poisson measure), and this was proved recently by Bollobás and Riordan [42] using the threshold Theorem 4.78 that is consequent on Theorem 4.26.

Bollobás and Riordan showed also in [43] that a similar argument leads to an approach to the proof that the critical probability of bond percolation on $\mathbb{Z}^2$ equals $\frac{1}{2}$. They used Theorem 4.78 in place of Kesten's explicit proof of sharp threshold for this model, see [135, 136]. A "shorter" version of [43] is presented in Section 5.8 for the case of site percolation on the triangular lattice.

We return to the influence theorem and its ramifications. There are several useful references concerning influence for product measures, see [84, 85, 130, 132] and their bibliographies[2]. The order of magnitude $N^{-1}\log N$ is the best possible in (4.28), as shown by the following 'tribes' example taken from [27]. A population of N individuals comprises t 'tribes' each of cardinality $s = \log N - \log\log N + \alpha$. Each individual votes 1 with probability $\frac{1}{2}$ and otherwise 0, and different individuals vote independently of one another. Let A be the event that there exists a tribe all of whose members vote 1. It is easily seen that

$$\begin{aligned} 1 - P(A) &= \left(1 - \frac{1}{2^s}\right)^t \\ &\sim e^{-t/2^s} \sim e^{-1/2^\alpha}, \end{aligned}$$

[2]The treatment presented here makes heavy use of the work of the 'Israeli' school. The earlier paper of Russo [188] must not be overlooked, and there are several important papers of Talagrand [203, 204, 205, 206]. Later approaches to Theorem 4.26 can be found in [77, 183, 184].

and, for all i,

$$I_A(i) = \left(1 - \frac{1}{2^s}\right)^{t-1} \frac{1}{2^{s-1}}$$
$$\sim e^{-1/2^\alpha} 2^{\alpha-1} \frac{\log N}{N},$$

The 'basic' Theorem 4.26 on the discrete cube $\Omega = \{0,1\}^E$ can be extended to the 'continuum' cube $K = [0,1]^E$, and hence to other product spaces. We state the result for K next. Let λ be uniform (Lebesgue) measure on K. For a measurable subset $A \subseteq K$, it is usual (see, for example, [46]) to define the influence of $e \in E$ on A as

$$L_A(e) = \lambda_{N-1}\big(\{\omega \in K : 1_A(\omega) \text{ is a non-constant function of } \omega(e)\}\big).$$

That is, $L_A(e)$ is the $(N-1)$-dimensional Lebesgue measure of the set of all $\psi \in [0,1]^{E\setminus\{e\}}$ with the property that: both A and its complement $\overline{A}$ intersect the 'fibre'

$$F_\psi = \{\psi\} \times [0,1] = \{\omega \in K : \omega(f) = \psi(f),\ f \neq e\}.$$

It is more natural to consider elements ψ for which $A \cap F_\psi$ has Lebesgue measure strictly between 0 and 1, and thus we define the influence in these notes by

$$I_A(e) = \lambda_{N-1}\big(\{\psi \in [0,1]^{E\setminus\{e\}} : 0 < \lambda_1(A \cap F_\psi) < 1\}\big). \tag{4.29}$$

Here and later, we write λ_k for k-dimensional Lebesgue measure. Note that $I_A(e) \leq L_A(e)$.

(4.30) Theorem [46]. *There exists a constant $c \in (0,\infty)$ such that the following holds. Let $N \geq 1$, let E be a finite set with $|E| = N$, and let A be an increasing subset of the cube $K = [0,1]^E$ with $\lambda(A) \in (0,1)$. Then*

$$\sum_{e \in E} I_A(e) \geq c\lambda(A)(1-\lambda(A)) \log[1/(2m)], \tag{4.31}$$

where $m = \max_e I_A(e)$, and the reference measure on K is Lebesgue measure λ. There exists $e \in E$ such that

$$I_A(e) \geq c\lambda(A)(1-\lambda(A)) \frac{\log N}{N}. \tag{4.32}$$

We shall see in Theorem 4.35 that the condition of monotonicity of A can be removed. The factor '2' in (4.31) is innocent in the following regard. The inequality is important only when m is small, and, for $m \leq \frac{1}{3}$ say, one may remove the '2' and replace c by a larger constant.

Results similar to those of Theorems 4.26 and 4.30 have been proved in [89] for certain non-product measures, and all increasing events. Let μ be a positive probability measure on the discrete space $\Omega = \{0, 1\}^E$ satisfying the FKG lattice condition (4.12). For any increasing subset A of Ω with $\mu(A) \in (0, 1)$, we have that

$$\sum_{e\in E} J_A(e) \ge c\mu(A)(1 - \mu(A)) \log[1/(2m)], \tag{4.33}$$

where $m = \max_e J_A(e)$. Furthermore, as above, there exists $e \in E$ such that

$$J_A(e) \ge c\mu(A)(1 - \mu(A)) \frac{\log N}{N}. \tag{4.34}$$

Note the use of *conditional* influence $J_A(e)$, with reference measure μ. Indeed, (4.34) can fail for all e when J_A is replaced by I_A. The proof of (4.33) makes use of Theorem 4.30, and is omitted here, see [89, 90].

The domain of Theorem 4.30 can be extended to powers of an arbitrary probability space, that is with $([0, 1], \lambda_1)$ replaced by a general probability space. Let $|E| = N$ and let $X = (\Sigma, \mathcal{F}, \mathbf{P})$ be a probability space. We write X^E for the product space of X. Let $A \subseteq \Sigma^E$ be measurable. The influence of $e \in E$ is given as in (4.29) by

$$I_A(e) = P\bigl(\{\psi \in \Sigma^{E\setminus\{e\}} : 0 < \mathbf{P}(A \cap F_\psi) < 1\}\bigr),$$

with $P = \mathbf{P}^E$ and $F_\psi = \{\psi\}\times\Sigma$, the 'fibre' of all $\omega \in X^E$ such that $\omega(f) = \psi(f)$ for $f \ne e$.

The following theorem contains two statements: that the influence inequalities are valid for general product spaces, and that they hold for non-increasing events. We shall require a condition on $X = (\Sigma, \mathcal{F}, \mathbf{P})$ for the first of these, and we state this next[3]. The pair $(\mathcal{F}, \mathbf{P})$ generates a measure ring (see [113, §40] for the relevant definitions). We call this measure ring *separable* if it is separable when viewed as a metric space with metric $\rho(B, B') = \mathbf{P}(B \triangle B')$.

(4.35) Theorem [46]. *Let $X = (\Sigma, \mathcal{F}, \mathbf{P})$ be a probability space whose non-atomic part is separable. Let $N \ge 1$, let E be a finite set with $|E| = N$, and let $A \subseteq \Sigma^E$ be measurable in the product space X^E, with $P(A) \in (0, 1)$. There exists an absolute constant $c \in (0, \infty)$ such that:*

$$\sum_{e\in E} I_A(e) \ge cP(A)(1 - P(A)) \log[1/(2m)], \tag{4.36}$$

where $m = \max_e I_A(e)$, and the reference measure is $P = \mathbf{P}^E$. There exists $e \in E$ with

$$I_A(e) \ge cP(A)(1 - P(A)) \frac{\log N}{N}. \tag{4.37}$$

Of especial interest is the case when $\Sigma = \{0, 1\}$ and $\mathbf{P}$ is Bernoulli measure with density p. Note that the atomic part of X is always separable, since there can be at most countably many atoms.

[3]This condition is omitted from [46].

4.6 Proofs of influence theorems

This section contains the proofs of the theorems of the last.

Proof of Theorem 4.26. We use a Fourier analysis of functions $f : \Omega \to \mathbb{R}$. Define the inner product by

$$\langle f, g \rangle = \phi(fg), \qquad f, g : \Omega \to \mathbb{R},$$

where $\phi = \phi_{\frac{1}{2}}$, so that the L^2-norm of f is given by

$$\| f \|_2 = \sqrt{\phi(f^2)} = \sqrt{\langle f, f \rangle}.$$

We call f *Boolean* if it takes values in the set $\{0, 1\}$. Boolean functions are in one–one correspondence with the power set of E via the relation $f = 1_A \leftrightarrow A$. If f is Boolean, say $f = 1_A$, then

$$\| f \|_2^2 = \phi(f^2) = \phi(f) = \phi(A). \tag{4.38}$$

For $F \subseteq E$, let

$$u_F(\omega) = \prod_{e \in F} (-1)^{\omega(e)} = (-1)^{\sum_{e \in F} \omega(e)}, \qquad \omega \in \Omega.$$

It can be checked that the functions u_F, $F \subseteq E$, form an orthonormal basis for the function space. Thus, for $f : \Omega \to \mathbb{R}$,

$$f = \sum_{F \subseteq E} \hat{f}(F) u_F,$$

where the so-called Fourier–Walsh coefficients of f are given by

$$\hat{f}(F) = \langle f, u_F \rangle, \qquad F \subseteq E.$$

In particular,

$$\hat{f}(\varnothing) = \phi(f),$$

and

$$\langle f, g \rangle = \sum_{F \subseteq E} \hat{f}(F) \hat{g}(F),$$

and the latter yields the Parseval relation

$$\| f \|_2^2 = \sum_{F \subseteq E} \hat{f}(F)^2. \tag{4.39}$$

Fourier analysis operates harmoniously with influences as follows. For $f = 1_A$ and $e \in E$, let

$$f_e(\omega) = f(\omega) - f(\kappa_e\omega),$$

where $\kappa_e\omega$ is the configuration ω with the state of e flipped. Since f_e takes values in the set $\{-1, 0, +1\}$, we have that $|f_e| = f_e^2$. The Fourier–Walsh coefficients of f_e are given by

$$\begin{aligned}\hat{f}_e(F) = \langle f_e, u_F\rangle &= \sum_{\omega\in\Omega} \frac{1}{2^N}\big[f(\omega) - f(\kappa_e\omega)\big](-1)^{|B\cap F|}\\ &= \sum_{\omega\in\Omega} \frac{1}{2^N} f(\omega)\big[(-1)^{|B\cap F|} - (-1)^{|(B\,\triangle\,\{e\})\cap F|}\big],\end{aligned}$$

where $B = \eta(\omega) := \{e \in E : \omega(e) = 1\}$ is the set of ω-open indices. Now,

$$\big[(-1)^{|B\cap F|} - (-1)^{|(B\,\triangle\,\{e\})\cap F|}\big] = \begin{cases} 0 & \text{if } e \notin F,\\ 2(-1)^{|B\cap F|} = 2u_F(\omega) & \text{if } e \in F,\end{cases}$$

so that

$$\hat{f}_e(F) = \begin{cases} 0 & \text{if } e \notin F,\\ 2\hat{f}(F) & \text{if } e \in F.\end{cases} \tag{4.40}$$

The influence $I(e) = I_A(e)$ is the mean of $|f_e| = f_e^2$, whence, by (4.39),

$$I(e) = \|f_e\|_2^2 = 4\sum_{F:\,e\in F} \hat{f}(F)^2, \tag{4.41}$$

and the total influence is

$$\sum_{e\in E} I(e) = 4\sum_{F\subseteq E} |F|\,\hat{f}(F)^2. \tag{4.42}$$

We propose to find an upper bound for the sum $\phi(A) = \sum_F \hat{f}(F)^2$. From (4.42) we will extract an upper bound for the contributions to this sum from the $\hat{f}(F)^2$ for large $|F|$. This will be combined with a corresponding estimate for small $|F|$ that will be obtained as follows by considering a re-weighted sum $\sum_F \hat{f}(F)^2\rho^{2|F|}$ for $0 < \rho < 1$.

For $w \in [1,\infty)$, we define the L^w-norm

$$\|g\|_w = \phi(|g|^w)^{1/w}, \qquad g : \Omega \to \mathbb{R},$$

recalling that $\|g\|_w$ is non-decreasing in w. For $\rho \in \mathbb{R}$, let $T_\rho g$ be the function

$$T_\rho g = \sum_{F\subseteq E} \hat{g}(F)\rho^{|F|}u_F$$

so that

$$\|T_\rho g\|_2^2 = \sum_{F\subseteq E} \hat{g}(F)^2 \rho^{2|F|}.$$

When $\rho \in [-1, 1]$, $T_\rho g$ has a probabilistic interpretation. For $\omega \in \Omega$, let $\Psi = (\Psi(e) : e \in E)$ be a random vector such that: the $\Psi(e)$, $e \in E$, are independent, and

$$\Psi(e) = \begin{cases} \omega(e) & \text{with probability } \frac{1}{2}(1+\rho), \\ 1-\omega(e) & \text{otherwise.} \end{cases}$$

We claim that

$$T_\rho g(\omega) = \mathbb{E}(g(\Psi)), \tag{4.43}$$

thus explaining why T_ρ is sometimes called the 'noise operator'. Equation (4.43) is proved as follows. First, for $F \subseteq E$,

$$\begin{aligned} \mathbb{E}(u_F(\Psi)) &= \mathbb{E}\Bigl(\prod_{e\in F}(-1)^{\Psi(e)}\Bigr) \\ &= \prod_{e\in F}(-1)^{\omega(e)}\bigl[\tfrac{1}{2}(1+\rho) - \tfrac{1}{2}(1-\rho)\bigr] \\ &= \rho^{|F|}u_F(\omega). \end{aligned}$$

Now, $g = \sum_F \hat{g}(F)u_F$, so that

$$\begin{aligned} \mathbb{E}(g(\Psi)) &= \sum_{F\subseteq E} \hat{g}(F)\mathbb{E}(u_F(\Psi)) \\ &= \sum_{F\subseteq E} \hat{g}(F)\rho^{|F|}u_F(\omega) = T_\rho g(\omega), \end{aligned}$$

as claimed at (4.43).

The next proposition is pivotal for the proof of the theorem. It is sometimes referred to as the 'hypercontractivity' lemma, and it is related to the log-Sobolev inequality. It is commonly attributed to subsets of Bonami [45], Gross [112], Beckner [24], each of whom has worked on estimates of this type. The proof is omitted.

(4.44) Proposition. *For $g : \Omega \to \mathbb{R}$ and $\rho > 0$,*

$$\|T_\rho g\|_2 \le \|g\|_{1+\rho^2}.$$

Let $0 < \rho < 1$. Set $g = f_e$ where $f = 1_A$, noting that g takes the values 0, ± 1 only. Then,

$$\begin{aligned}\sum_{F:e\in F} 4\hat{f}(F)^2\rho^{2|F|} &= \sum_{F\subseteq E} \hat{f}_e(F)^2\rho^{2|F|} && \text{by (4.40)}\\ &= \|T_\rho f_e\|_2^2 \\ &\le \|f_e\|_{1+\rho^2}^2 = \left[\phi(|f_e|^{1+\rho^2})\right]^{2/(1+\rho^2)} && \text{by Proposition 4.44}\\ &= \|f_e\|_2^{4/(1+\rho^2)} = I(e)^{2/(1+\rho^2)} && \text{by (4.41).}\end{aligned}$$

Therefore,

$$\sum_{e\in E} I(e)^{2/(1+\rho^2)} \ge 4\sum_{F\subseteq E} |F|\,\hat{f}(F)^2\rho^{2|F|}. \tag{4.45}$$

Let $t = \phi(A) = \hat{f}(\varnothing)$. By (4.45),

$$\begin{aligned}\sum_{e\in E} I(e)^{2/(1+\rho^2)} &\ge 4\rho^{2b}\sum_{0<|F|\le b} \hat{f}(F)^2 \\ &= 4\rho^{2b}\Bigl(\sum_{|F|\le b} \hat{f}(F)^2 - t^2\Bigr),\end{aligned} \tag{4.46}$$

where $b \in (0,\infty)$ will be chosen later. By (4.42),

$$\sum_{e\in E} I(e) \ge 4b\sum_{|F|>b} \hat{f}(F)^2,$$

which we add to (4.46) to obtain

$$\begin{aligned}\rho^{-2b}\sum_{e\in E} I(e)^{2/(1+\rho^2)} + \frac{1}{b}\sum_{e\in E} I(e) &\ge 4\sum_{F\subseteq E}\hat{f}(F)^2 - 4t^2 \\ &= 4t(1-t) \qquad \text{by (4.39).}\end{aligned} \tag{4.47}$$

We are now ready to prove (4.27). Let $m = \max_e I(e)$, noting that $m > 0$ since $\phi(A) \ne 0, 1$. The claim is trivial if $m = 1$, and we assume that $m < 1$. Then

$$\sum_{e\in E} I(e)^{4/3} \le m^{1/3}\sum_{e\in E} I(e),$$

whence, by (4.47) and the choice $\rho^2 = \frac{1}{2}$,

$$\left(2^b m^{1/3} + \frac{1}{b}\right)\sum_{e\in E} I(e) \ge 4t(1-t). \tag{4.48}$$

We choose b such that $2^b m^{1/3} = b^{-1}$, and it is an easy exercise that $b \geq A\log(1/m)$ for some absolute constant $A > 0$. With this choice of b, (4.27) follows from (4.48) with $c = 2A$. Inequality (4.32) follows, as explained after the statement of the theorem. □

Proof of Theorem 4.30. We follow [84]. The idea of the proof is to 'discretize' the cube K and the increasing event A, and to apply Theorem 4.26.

Let $k \in \{1, 2, \dots\}$ to be chosen later, and subdivide the N-cube $K = [0,1]^E$ into 2^{kN} disjoint smaller cubes each of side-length 2^{-k}. These small cubes are of the form

$$B(\mathbf{l}) = \prod_{e \in E} [l_e, l_e + 2^{-k}), \tag{4.49}$$

where $\mathbf{l} = (l_e : e \in E)$ and each l_e is a 'binary decimal' of the form $l_e = 0.l_{e,1}l_{e,2}\cdots l_{e,k}$ with each $l_{e,j} \in \{0,1\}$. There is a special case. When $l_e = 0.11\cdots 1$, we put the closed interval $[l_e, l_e + 2^{-k}]$ into the product of (4.49). Lebesgue measure λ on K induces product measure ϕ with density $\frac{1}{2}$ on the space $\Omega = \{0,1\}^{kN}$ of 0/1-vectors $(l_{e,j} : j = 1, 2, \dots, k,\ e \in E)$. We call each $B(\mathbf{l})$ a 'small cube'.

We claim that it suffices to consider events A that are the unions of small cubes. For a measurable subset $A \subseteq K$, let $\hat{A}$ be the subset of K that 'approximates' to A, given by $\hat{A} = \bigcup_{\mathbf{l} \in \mathcal{A}} B(\mathbf{l})$ where

$$\mathcal{A} = \{\mathbf{l} \in \Omega : B(\mathbf{l}) \cap A \neq \varnothing\}.$$

Note that $\mathcal{A}$ is an increasing subset of the discrete kN-cube Ω. We write $I_{\mathcal{A}}(e, j)$ for the influence of the index (e, j) on the subset $\mathcal{A} \subseteq \Omega$ under the measure ϕ. The next task is to show that, when replacing A by $\hat{A}$, the measure and influences of A are not greatly changed.

(4.50) Lemma [46]. *In the above notation,*

$$0 \leq \lambda(\hat{A}) - \lambda(A) \leq \frac{N}{2^k}, \tag{4.51}$$

$$|I_{\hat{A}}(e) - I_A(e)| \leq \frac{2N}{2^k}, \qquad e \in E. \tag{4.52}$$

Proof. Clearly $A \subseteq \hat{A}$, whence $\lambda(A) \leq \lambda(\hat{A})$. Let $\mu : K \to K$ be the projection mapping that maps $(x_f : f \in E)$ to $(x_f - m : f \in E)$ where $m = \min_{g \in E} x_g$. We have that

$$\lambda(\hat{A}) - \lambda(A) \leq |R| 2^{-kN}, \tag{4.53}$$

where R is the set of small cubes that intersect both A and its complement $\overline{A}$. Since A is increasing, R cannot contain two distinct elements r, r' with $\mu(r) = \mu(r')$.

Therefore, $|R|$ is no larger than the number of faces of small cubes lying in the 'hyperfaces' of K, that is,

$$K \le N 2^{k(N-1)}. \tag{4.54}$$

Inequality (4.51) follows by (4.53).

Let $e \in E$, and let $\mu_e : K \to [0,1]^{E \setminus \{e\}}$ be the projection that sends $(x_f : f \in E)$ to $(x_f : f \in E \setminus \{e\})$. The face $\mu_e(K)$ is the union of 'small faces' of small cubes. Each small face F corresponds to a 'tube' $T(F)$ of small cubes, based on that face with axis parallel to the eth direction. See Figure 4.1. Such a tube has 'first' face F and 'last' face $L = T(F) \cap \{\omega \in K : \omega(e) = 1\}$, and we write B_F (respectively, B_L) for the (unique) small cube with face F (respectively, L).

It is easily seen that F contributes 0 to $I_{\hat{A}}(e) - I_A(e)$ if 1_A is constant on both B_F and B_L (it is not important that 1_A should take the same value on the initial small cube as on the final). Therefore,

$$|I_{\hat{A}}(e) - I_A(e)| \le |N_F \cup N_L| 2^{-k(N-1)}, \tag{4.55}$$

where N_F (respectively, N_L) is the set of initial (respectively, final) small cubes on which 1_A is non-constant. By restricting 1_A to the 'fattened hyperface' $\bigcup \{B_F : F \subseteq \mu_e(K)\}$ and applying the argument leading to (4.54) within this region, we find as there that

$$|N_F| \le (N-1) 2^{k(N-2)}.$$

The same inequality holds with N_L in place of N_F, and inequality (4.52) follows by (4.55). □

Let A be an increasing subset of K, assume $0 < t = \lambda(A) < 1$, and let $m = \max_e I_A(e)$. We may assume that $0 < m < \frac{1}{2}$, since otherwise (4.31) is a triviality. With $\hat{A}$ given as above for some value of k to be chosen soon, we write $\hat{t} = \lambda(\hat{A})$ and $\hat{m} = \max_e I_{\hat{A}}(e)$. We shall prove below that

$$\sum_{e \in E} I_{\hat{A}}(e) \ge c \hat{t}(1-\hat{t}) \log[1/(2\hat{m})], \tag{4.56}$$

for some absolute constant $c > 0$. Let $k = k(N, A)$ be sufficiently large that the following inequalities hold:

$$\frac{N}{2^k} \le \frac{1}{2} \min \left\{ t(1-t), \frac{m \log[1/(2m)]}{2 + \log[1/(2m)]}, \frac{1}{2} - m \right\}, \tag{4.57}$$

$$\frac{2N^2}{2^k} \le \frac{1}{8} c t(1-t) \log[1/(2m)]. \tag{4.58}$$

By Lemma 4.50,

$$|t - \hat{t}| \le \frac{N}{2^k}, \qquad |m - \hat{m}| \le \frac{2N}{2^k}, \tag{4.59}$$

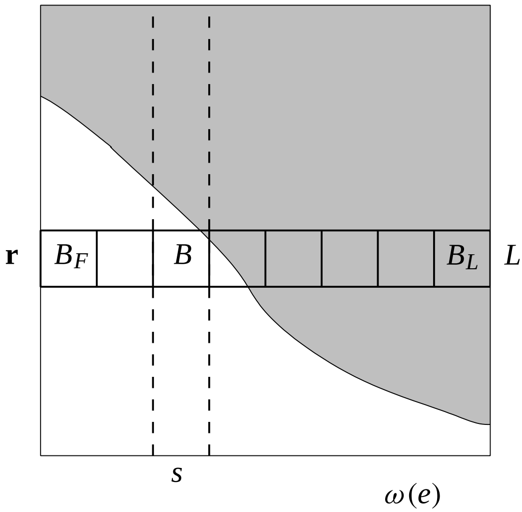

Figure 4.1. The small boxes $B = B(\mathbf{r}, s)$ form the tube $T(\mathbf{r})$. The region A is shaded.

whence, by (4.57)–(4.58),
(4.60)

$$|t - \hat{t}| \le \tfrac{1}{2}t(1-t), \quad \hat{m} < \tfrac{1}{2}, \quad \frac{|m - \hat{m}|}{m \wedge \hat{m}} \le \tfrac{1}{2}\log[1/(2m)].$$

By Lemma 4.50 again,

$$\sum_{e \in A} I_A(e) \ge \sum_{e \in A} I_{\hat{A}}(e) - \frac{2N^2}{2^k}.$$

By (4.56), (4.58), and (4.60),

$$\begin{aligned}\sum_{e \in A} I_A(e) &\ge c\big[t(1-t) - |t - \hat{t}|\big]\left[\log[1/(2m)] - \frac{|m-\hat{m}|}{m \wedge \hat{m}}\right] - \frac{2N^2}{2^k} \\ &\ge \tfrac{1}{8}ct(1-t)\log[1/(2m)]\end{aligned}$$

as required.

It thus suffices to prove (4.56), and *we shall henceforth assume that A is a union of small cubes*.

(4.61) Lemma [46, 84]. *For $e \in E$,*

$$\sum_{j=1}^{k} I_{\mathcal{A}}(e, j) \le 2I_A(e).$$

Proof. Let $e \in E$. For a fixed vector $\mathbf{r} = (r_1, r_2, \ldots, r_{N-1}) \in (\{0,1\}^k)^{E\setminus\{e\}}$, consider the 'tube' $T(\mathbf{r})$ comprising the union of the small cubes $B(\mathbf{r}, s)$ over the 2^k possible values in $s \in \{0,1\}^k$. One sees after a little thought (see Figure 4.1) that

$$I_{\mathcal{A}}(e, j) = \sum_{\mathbf{r}} (\tfrac{1}{2})^{kN-1} K(\mathbf{r}, j),$$

where $K(\mathbf{r}, j)$ is the number of unordered pairs $S = B(\mathbf{r}, s)$, $S' = B(\mathbf{r}, s')$ of small cubes of $T(\mathbf{r})$ such that: $S \subseteq A$, $S' \not\subseteq A$, and $|s - s'| = 2^{-j}$. Since A is an increasing subset of K, one can see that

$$K(\mathbf{r}, j) \le 2^{k-j}, \qquad j = 1, 2, \ldots, k,$$

whence

$$\sum_j I_{\mathcal{A}}(e, j) \le \frac{2^k}{2^{kN-1}} J_N = \frac{2}{2^{k(N-1)}} J_N,$$

where J_N is the number of tubes $T(\mathbf{r})$ that intersect both A and its complement $\overline{A}$. Now,

$$I_A(e) = \frac{1}{2^{k(N-1)}} J_N,$$

and the lemma is proved. □

We return to the proof of (4.56). The c_j that follow are absolute positive constants. Assume that $m = \max_e I_A(e) < \frac{1}{2}$. By Lemma 4.61,

$$I_{\mathcal{A}}(e, j) \le 2m \qquad \text{for all } e, j.$$

By (4.27) applied to the event $\mathcal{A}$ of the kN-cube Ω,

$$\sum_{e,j} I_{\mathcal{A}}(e, j) \ge c_2 t(1-t)\log[1/(2m)],$$

where $t = \lambda(A)$. By Lemma 4.61 again,

$$\sum_{e \in E} I_A(e) \ge \tfrac{1}{2} c_2 t(1-t)\log[1/(2m)],$$

as required at (4.56). □

Proof of Theorem 4.35. We prove this in two steps.

I. In the notation of the theorem, there exists a Lebesgue-measurable subset B of $K = [0,1]^E$ such that: $P(A) = \lambda(B)$, and $I_A(e) \ge I_B(e)$ for all e, where the influences are calculated according to the appropriate probability measures.

II. There exists an increasing subset C of K such that $\lambda(B) = \lambda(C)$, and $I_B(e) \ge I_C(e)$ for all e.

The claims of the theorem follow via Theorem 4.30 from these two facts.

A version of Claim I was stated in [46] without proof. We use the measure-space isomorphism theorem[4], Theorem B of [113, p. 173] (see also [1, p. 3] or [174, p. 16]). Let $x_1, x_2, \dots$ be an ordering of the atoms of X, and let I_i be the sub-interval $[q_i, q_{i+1})$ of $[0, 1]$, where

$$q_i = \sum_{j=1}^{i-1} \mathbf{P}(\{x_j\}).$$

The non-atomic part of X has sample space $\Sigma' = \Sigma \setminus \{x_1, x_2, \dots\}$, and total measure $1 - q_\infty$. By the isomorphism theorem, there exists a measure-preserving map μ from the σ-algebra $\mathcal{F}'$ of Σ' to the Borel σ-algebra of the interval $[q_\infty, 1]$ endowed with Lebesgue measure λ_1, satisfying

$$\begin{aligned} \mu(A_1 \setminus A_2) &\stackrel{\lambda}{=} \mu A_1 \setminus \mu A_2, \\ \mu\left(\bigcup_{n=1}^{\infty} A_n\right) &\stackrel{\lambda}{=} \bigcup_{n=1}^{\infty} \mu A_n, \end{aligned} \tag{4.62}$$

for $A_n \in \mathcal{F}'$, where $A \stackrel{\lambda}{=} B$ means that $\lambda_1(A \triangle B) = 0$. We extend the domain of μ to $\mathcal{F}$ by setting $\mu(\{x_i\}) = I_i$. In summary, there exists $\mu : \mathcal{F} \to \mathcal{B}[0, 1]$ such that $\mathbf{P}(A) = \lambda_1(\mu A)$ for $A \in \mathcal{F}$, and (4.62) holds for $A_n \in \mathcal{F}$.

The product σ-algebra $\mathcal{F}^E$ of X^E is generated by the class $\mathcal{R}^E$ of 'rectangles' of the form $R = \prod_{e \in E} A_e$ for $A_e \in \mathcal{F}$. For such $R \in \mathcal{R}^E$, let

$$\mu^E R = \prod_{e \in E} \mu A_e.$$

We extend the domain of μ^E to the class $\mathcal{U}$ of finite unions of rectangles by

$$\mu^E\left(\bigcup_{i=1}^{m} R_i\right) = \bigcup_{i=1}^{m} \mu^E R_i.$$

It can be checked that

$$P(R) = \lambda^E(\mu^E R), \tag{4.63}$$

for any such union R.

Let $A \in \mathcal{F}^E$. We can find an increasing sequence $(U_n : n \geq 1)$ of elements of $\mathcal{U}$, each being a union of rectangles with strictly positive measure, such that $P(A \triangle U_n) \to 0$ as $n \to \infty$, and hence

$$P(U_n \setminus A) = 0. \tag{4.64}$$

[4] Tom Liggett kindly proposed the use of the isomorphism theorem.

Let $V_n = \mu^E U_n$ and $B = \lim_{n\to\infty} V_n$. Since V_n is non-decreasing in n, by (4.63),

$$\lambda^E(B) = \lim_{n\to\infty} \lambda^E(\mu^E U_n) = \lim_{n\to\infty} P(U_n) = P(A).$$

We turn now to the influences. Let $e \in E$, and

$$J_A^a = \mathbf{P}^{E\setminus\{e\}}\big(\{\psi \in \Sigma^{E\setminus\{e\}} : \mathbf{P}(A \cap F_\psi) = a\}\big), \qquad a = 0, 1,$$

where $F_\psi = \{\psi\} \times \Sigma$ is the 'fibre' at ψ. We define J_B^a similarly, with $\mathbf{P}$ replaced by λ and F_ψ replaced by the fibre $\{\psi\} \times [0, 1]$ of K. Thus,

$$I_A(e) = 1 - J_A^0 - J_A^1, \tag{4.65}$$

and we claim that

$$J_A^0 \le J_B^0. \tag{4.66}$$

By replacing A by its complement $\overline{A}$, we obtain that $J_A^1 \le J_B^1$, and it follows by (4.65)–(4.66) that $I_A(e) \ge I_B(e)$, as required. We write U_n as the finite union $U_n = \bigcup_i F_i \times G_i$ where each F_i (respectively, G_i) is a rectangle of $\Sigma^{E\setminus\{e\}}$ (respectively, Σ). By Fubini's theorem and (4.64),

$$\begin{aligned} J_A^0 \le J_{U_n}^0 &= 1 - \mathbf{P}^{E\setminus\{e\}}\Big(\bigcup_i F_i\Big) \\ &= 1 - \lambda^{E\setminus\{e\}}\Big(\bigcup_i \mu^{E\setminus\{e\}} F_i\Big) = J_{V_n}^0, \end{aligned}$$

by (4.63) with E replaced by $E \setminus \{e\}$.

Finally, we show that $J_{V_n}^0 \to J_B^0$ as $n \to \infty$, and (4.66) will follow. For $\psi \in \Sigma^E$, we write $\mathrm{proj}(\psi)$ for the projection of ψ onto the sub-space $\Sigma^{E\setminus\{e\}}$. Since the V_n are unions of rectangles of $[0, 1]^E$ with strictly positive measure,

$$J_{V_n}^0 = \lambda^{E\setminus\{e\}}\big(\mathrm{proj}\, \overline{V_n}\big).$$

Now, $V_n \uparrow B$, so that $\mathrm{proj}\, \overline{V_n} \downarrow \mathrm{proj}\, \overline{B}$, whence $J_{V_n}^0 \to \lambda^{E\setminus\{e\}}(\mathrm{proj}\, \overline{B})$. Using the fact that $\psi \in B$ if and only if $\psi \in V_n$ for some n, we have that $\lambda^{E\setminus\{e\}}(\mathrm{proj}\, \overline{B}) = J_B^0$, and (4.66) follows. Claim I is proved.

Claim II is proved by an elaboration of the method laid out in [27, 46]. Let $B \subseteq K$ be a non-increasing event. For $e \in E$ and $\psi = (\omega(g) : g \ne e) \in [0, 1]^{E\setminus\{e\}}$, we define the fibre F_ψ as usual by $F_\psi = \{\psi\} \times [0, 1]$. We replace $B \cap F_\psi$ by the set

$$B_\psi = \begin{cases} \{\psi\} \times (1 - y, 1] & \text{if } y > 0, \\ \varnothing & \text{if } y = 0, \end{cases} \tag{4.67}$$

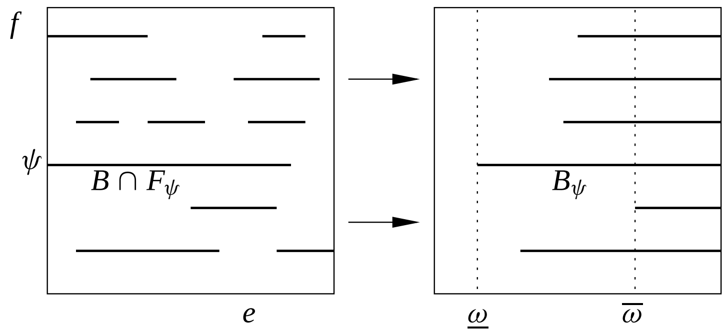

Figure 4.2. In the e/f-plane, we push every $B \cap F_\psi$ as far rightwards along the fibre F_ψ as possible.

where

$$y = y(\psi) = \lambda_1(B \cap F_\psi). \tag{4.68}$$

Thus B_ψ is obtained from B by 'pushing $B \cap F_\psi$ up the fibre' in a measure-preserving manner. See Figure 4.2. Clearly, $M_e B = \bigcup_\psi B_\psi$ is increasing[5] in the direction e and, by Fubini's theorem,

$$\lambda(M_e B) = \lambda(B). \tag{4.69}$$

We order E in an arbitrary manner, and let

$$C = \left(\prod_{e \in E} M_e \right) B,$$

where the product is constructed in the given order. By (4.69), $\lambda(C) = \lambda(B)$. We show that C is increasing by proving that: if B is increasing in direction $f \in E$ where $f \neq e$, then so is $M_e B$. It is enough to work with the reduced sample space $K' = [0, 1]^{\{e, f\}}$, as illustrated in Figure 4.2. Suppose that $\omega, \omega' \in K'$ are such that $\omega(e) = \omega'(e)$ and $\omega(f) < \omega'(f)$. Then

$$1_{M_e B}(\omega) = \begin{cases} 1 & \text{if } \omega(e) > 1 - y, \\ 0 & \text{if } \omega(e) \le 1 - y, \end{cases} \tag{4.70}$$

where $y = y(\omega(f))$ is given according to (4.68), with a similar expression with ω and y replaced by ω' and y'. Since B is assumed increasing in $\omega(f)$, we have that $y \le y'$. By (4.70), if $\omega \in M_e B$, then $\omega' \in M_e B$, which is to say that $M_e B$ is increasing in direction f.

Finally, we show that

$$I_{M_e B}(f) \le I_B(f), \qquad f \in E, \tag{4.71}$$

[5] *Exercise*: Show that $M_e B$ is Lebesgue-measurable.

whence $I_C(f) \le I_B(f)$ and the theorem is proved. First, by construction, $I_{M_e B}(e) = I_B(e)$. Let $f \ne e$. By conditioning on $\omega(g)$ for $g \ne e, f$,

$$I_{M_e B}(f) = \lambda^{E \setminus \{e,f\}}\Big(\lambda_1\big(\{\omega(e) : 0 < \lambda_1(M_e B \cap F_\nu) < 1\}\big)\Big)$$

where $\nu = (\omega(g) : g \ne f)$ and $F_\nu = \{\nu\} \times [0, 1]$. We shall show that

(4.72)
$$\lambda_1\big(\{\omega(e) : 0 < \lambda_1(M_e B \cap F_\nu) < 1\}\big) \le \lambda_1\big(\{\omega(e) : 0 < \lambda_1(B \cap F_\nu) < 1\}\big),$$

and the claim will follow. Inequality (4.72) depends only on $\omega(e)$, $\omega(f)$, and thus we shall make no further reference to the remaining coordinates $\omega(g)$, $g \ne e, f$. Henceforth, we write ω for $\omega(e)$ and ψ for $\omega(f)$.

With the aid of Figure 4.2, we see that the left side of (4.72) equals $\overline{\omega} - \underline{\omega}$, where

(4.73)
$$\overline{\omega} = \sup\{\omega : \lambda_1(M_e B \cap F_\omega) < 1\},$$
$$\underline{\omega} = \inf\{\omega : \lambda_1(M_e B \cap F_\omega) > 0\}.$$

Let ϵ be positive and small, and let

(4.74)
$$A_\epsilon = \{\psi : \lambda_1(B \cap F_\psi) > 1 - \underline{\omega} - \epsilon\}.$$

Since $\lambda_1(B \cap F_\psi) = \lambda_1(M_e B \cap F_\psi)$, $\lambda_1(A_\epsilon) > 0$ by (4.73). Let $A'_\epsilon = [0, 1] \times A_\epsilon$. We estimate the two-dimensional Lebesgue measure $\lambda_2(B \cap A'_\epsilon)$ in two ways:

$$\lambda_2(B \cap A'_\epsilon) > \lambda_1(A_\epsilon)(1 - \underline{\omega} - \epsilon) \qquad \text{by (4.74)},$$
$$\lambda_2(B \cap A'_\epsilon) \le \lambda_1(A_\epsilon)\lambda_1(\{\omega : \lambda_1(B \cap F_\omega) > 0\}),$$

whence $C = \{\omega : \lambda_1(B \cap F_\omega) > 0\}$ satisfies

$$\lambda_1(C) \ge \lim_{\epsilon \downarrow 0}[1 - \underline{\omega} - \epsilon] = 1 - \underline{\omega}.$$

By a similar argument, $D = \{\omega : \lambda_1(B \cap F_\omega) = 1\}$ satisfies

$$\lambda_1(D) \le 1 - \overline{\omega}.$$

For $\omega \in C \setminus D$, $0 < \lambda_1(B \cap F_\omega) < 1$, so that

$$I_B(e) \ge \lambda_1(C \setminus D) \ge \overline{\omega} - \underline{\omega},$$

and (4.71) follows. □

4.7 Russo formula, and sharp thresholds

Let ϕ_p denote product measure with density p on the finite product space $\Omega = \{0, 1\}^E$. The influence $I_A(e)$, of $e \in E$ on an event A, is given in (4.25).

(4.75) Theorem. Russo formula. *For any event $A \subseteq \Omega$,*

$$\frac{d}{dp}\phi_p(A) = \sum_{e\in E} [\phi_p(A^e) - \phi(A_e)] = \sum_{e\in E} I_A(e).$$

This formula, or its equivalent, has been discovered by a number of authors. See, for example, [23, 160, 187]. The element $e \in E$ is called *pivotal* for the event A if the occurrence or not of A depends on the state of e, that is, if $1_A(\omega_e) \neq 1_A(\omega^e)$. If A is increasing, Russo's formula states that $\phi_p'(A)$ equals the mean number of pivotal elements of E.

Proof. This is standard, see for example [95]. It is elementary that

$$\frac{d}{dp}\phi_p(A) = \sum_{\omega\in\Omega} \left(\frac{|\eta(\omega)|}{p} - \frac{N - |\eta(\omega)|}{1-p} \right) 1_A(\omega)\phi_p(\omega), \tag{4.76}$$

where $\eta(\omega) = \{e \in E : \omega(e) = 1\}$ and $N = |E|$. Setting $A = \Omega$, we find that

$$0 = \frac{1}{p(1-p)}\phi_p(|\eta| - pN),$$

whence

$$\begin{aligned} p(1-p)\frac{d}{dp}\phi_p(A) &= \phi_p\big([|\eta| - pN]1_A\big) - \phi_p(|\eta| - pN)\phi_p(1_A) \\ &= \phi_p(|\eta|1_A) - \phi_p(|\eta|)\phi_p(1_A) \\ &= \sum_{e\in E}\big[\phi_p(1_e 1_A) - \phi_p(1_e)\phi_p(1_A)\big], \end{aligned}$$

where 1_e is the indicator function that e is open. The summand equals

$$p\phi_p(A^e) - p\big[p\phi_p(A^e) + (1-p)\phi_p(A_e)\big].$$

and the formula is proved. □

Let A be an increasing event in $\Omega = \{0, 1\}^E$ that is non-trivial in that $A \neq \varnothing, \Omega$. The function $f(p) = \phi_p(A)$ is non-decreasing with $f(0) = 0$ and $f(1) = 1$. The next theorem is an immediate consequence of Theorems 4.35 and 4.75.

(4.77) Theorem [204]. *There exists a constant $c > 0$ such that the following holds. Let A be an increasing subset of Ω with $A \neq \varnothing, \Omega$. Then, for $p \in (0,1)$,*

$$\frac{d}{dp}\phi_p(A) \geq c\phi_p(A)(1-\phi_p(A)) \log[1/(2\max_e I_A(e)],$$

where $I_A(e)$ is the influence of e on A with respect to the measure ϕ_p.

Theorem 4.77 takes an especially simple form when A has a certain property of symmetry. In such a case, the following sharp-threshold theorem implies that $f(p) = \phi_p(A)$ increases from (near) 0 to (near) 1 over an interval of p-values with length of order not exceeding $1/\log N$.

Let Π be the group of permutations of E. Any $\pi \in \Pi$ acts on Ω by $\pi\omega = (\omega(\pi_e) : e \in E)$. We say that a subgroup $\mathcal{A}$ of Π *acts transitively* on E if, for all pairs $j, k \in E$, there exists $\alpha \in \mathcal{A}$ with $\alpha_j = k$.

Let $\mathcal{A}$ be a subgroup of Π. A probability measure ϕ on $(\Omega, \mathcal{F})$ is called $\mathcal{A}$-*invariant* if $\phi(\omega) = \phi(\alpha\omega)$ for all $\alpha \in \mathcal{A}$. An event $A \in \mathcal{F}$ is called $\mathcal{A}$-*invariant* if $A = \alpha A$ for all $\alpha \in \mathcal{A}$. It is easily seen that, for any subgroup $\mathcal{A}$, ϕ_p is $\mathcal{A}$-invariant.

(4.78) Theorem (Sharp threshold) [85]. *There exists a constant c satisfying $c \in (0,\infty)$ such that the following holds. Let $N = |E| \geq 1$. Let $A \in \mathcal{F}$ be an increasing event, and suppose there exists a subgroup $\mathcal{A}$ of Π acting transitively on E such that A is $\mathcal{A}$-invariant. Then*

$$\frac{d}{dp}\phi_p(A) \geq c\phi_p(A)(1-\phi_p(A)) \log N, \qquad p \in (0,1). \tag{4.79}$$

Let $\epsilon \in (0, \frac{1}{2})$ and let A be increasing and non-trivial (in the above sense). Under the conditions of the theorem, $\phi_p(A)$ increases from ϵ to $1-\epsilon$ over an interval of values of p having length of order not exceeding $1/\log N$. This amounts to a quantification of the so-called S-shape results described and cited in [95, Sect. 2.5]. An early step in the direction of sharp thresholds was taken by Russo [188] (see also [204]), but without the quantification of $\log N$.

Essentially the same conclusions hold for a family $\{\mu_p : p \in (0,1)\}$ of probability measures given as follows in terms of a positive measure μ satisfying the FKG lattice condition. For $p \in (0,1)$, let μ_p be given by

$$\mu_p(\omega) = \frac{1}{Z_p}\left(\prod_{e\in E} p^{\omega(e)}(1-p)^{1-\omega(e)}\right)\mu(\omega), \qquad \omega \in \Omega, \tag{4.80}$$

where Z_p is chosen in such a way that μ_p is a probability measure. It is easy to check that each μ_p satisfies the FKG lattice condition. It turns out that, for an increasing event $A \neq \varnothing, \Omega$,

$$\frac{d}{dp}\mu_p(A) \geq \frac{c\xi_p}{p(1-p)}\mu_p(A)(1-\mu_p(A)) \log[1/(2\max_e J_A(e)], \tag{4.81}$$

where

$$\xi_p = \min_{e \in E} \big[\mu_p(\omega(e) = 1)\mu_p(\omega(e) = 0)\big].$$

The proof uses inequality (4.33) and proceeds as in [89]. This extension of Theorem 4.77 does not appear to have been noted before. It may be used in the studies of the random-cluster model, and of the Ising model with external field (see [90]).

A slight variant of Theorem 4.78 is valid for measures ϕ_p given by (4.80), with the positive probability measure μ satisfying: μ satisfies the FKG lattice condition, and μ is $\mathcal{A}$-invariant. See (4.81) and [89, 98].

From amongst the issues arising from the sharp-threshold Theorem 4.78, we identify two. First, to what degree is information about the group $\mathcal{A}$ relevant to the sharpness of the threshold. Secondly, what can be said when $p = p_N$ tends to 0 as $N \to \infty$. The reader is referred to [132] for answers to these questions.

Proof of Theorem 4.78. We show first that the influences $I_A(e)$ are constant for $e \in E$. Let $e, f \in E$, and find $\alpha \in \mathcal{A}$ such that $\alpha_e = f$. Under the given conditions,

$$\begin{aligned}\phi_p(A,\ 1_f = 1) &= \sum_{\omega \in A} \phi_p(\omega) 1_f(\omega) = \sum_{\omega \in A} \phi_p(\alpha\omega) 1_e(\alpha\omega) \\ &= \sum_{\omega' \in A} \phi_p(\omega') 1_e(\omega') = \phi_p(A,\ 1_e = 1),\end{aligned}$$

where 1_g is the indicator function that $\omega(g) = 1$. On setting $A = \Omega$, we deduce that $\phi_p(1_f = 1) = \phi_p(1_e = 1)$. On dividing, we obtain that $\phi_p(A \mid 1_f = 1) = \phi_p(A \mid 1_e = 1)$. A similar equality holds with 1 replaced by 0, and therefore $I_A(e) = I_A(f)$.

It follows that

$$\sum_{f \in E} I_A(f) = N I_A(e).$$

By Theorem 4.35 applied to the product space $(\Omega, \mathcal{F}, \phi_p)$, the right side is at least $c\phi_p(A)(1 - \phi_p(A)) \log N$, and (4.79) is a consequence of Theorem 4.75. □

4.8 Exercises

4.1. Let $X_n, Y_n \in L^2(\Omega, \mathcal{F}, P)$ be such that $X_n \to X$, $Y_n \to Y$ in L^2. Show that $X_n Y_n \to XY$ in L^1. [Reminder: L^p is the set of random variables Z with $E(|Z|^p) < \infty$, and $Z_n \to Z$ in L^p if $E(|Z_n - Z|^p) \to 0$. You may use any standard fact such as the Cauchy–Schwarz inequality.]

4.2. [121] Let $\mathbb{P}_p$ be a product measure on the space $\{0, 1\}^n$ with density p. Show by induction on n that $\mathbb{P}_p$ satisfies the Harris–FKG inequality, which is to say that $\mathbb{P}_p(A \cap B) \geq \mathbb{P}_p(A)\mathbb{P}_p(B)$ for any pair A, B of increasing events.

4.3. (continuation) Consider bond percolation on the square lattice $\mathbb{Z}^2$. Let X and Y be increasing functions on the sample space, such that $\mathbb{P}_p(X^2), \mathbb{P}_p(Y^2) < \infty$. Show that X and Y are positively associated.

4.4. *Coupling.* (a) Take $\Omega = [0, 1]$, with the Borel σ-field and Lebesgue measure $\mathbb{P}$. For any distribution function F, define a random variable Z_F on Ω by

$$Z_F(\omega) = \inf\{z : \omega \le F(z)\}, \qquad \omega \in \Omega.$$

Prove that

$$\mathbb{P}(Z_F \le z) = \mathbb{P}\big([0, F(z)]\big) = F(z),$$

whence Z_F has distribution function F.

(b) For real-valued random variables X, Y, we write $X \le_{\text{st}} Y$ if $P(X \le u) \ge P(Y \le u)$ for all u. Show that $X \le_{\text{st}} Y$ if and only if there exist random variables X', Y' on Ω, with the same respective distributions as X and Y, such that $\mathbb{P}(X' \le Y') = 1$.

4.5. [98] Let μ be a positive probability measure on the finite product $\Omega = \{0, 1\}^E$. Show that it satisfies the FKG lattice condition

$$\mu(\omega_1 \vee \omega_2)\mu(\omega_1 \wedge \omega_2) \ge \mu(\omega_1)\mu(\omega_2), \qquad \omega_1, \omega_2 \in \Omega,$$

if and only if this inequality holds for all pairs ω_1, ω_2 that differ on exactly two elements of E.

4.6. [98] Let μ_1, μ_2 be positive probability measures on the finite product $\Omega = \{0, 1\}^E$. Assume that they satisfy

$$\mu_2(\omega_1 \vee \omega_2)\mu_1(\omega_1 \wedge \omega_2) \ge \mu_1(\omega_1)\mu_2(\omega_2),$$

for all pairs $\omega_1, \omega_2 \in \Omega$ that differ on either one or two elements of E, and in addition that μ_1 satisfies the FKG lattice condition. Show that $\mu_2 \ge_{\text{st}} \mu_1$.

4.7. Let $X_1, X_2, \dots$ be independent Bernoulli random variables with parameter p, and $S_n = X_1 + X_2 + \cdots + X_n$. Show by Hoeffding's inequality or otherwise that

$$P\big(|S_n - np| \ge x\sqrt{n}\big) \le 2\exp(-\tfrac{1}{2}x^2/m), \qquad x > 0,$$

where $m = \min\{p, 1 - p\}$.

4.8. Let $G_{n,p}$ be the random graph with vertex set $V = \{1, 2, \dots, n\}$ obtained by joining each pair of distinct vertices by an edge with probability p (different pairs are joined independently). Show that the chromatic number $\chi_{n,p}$ satisfies

$$P\big(|\chi_{n,p} - E\chi_{n,p}| \ge x\big) \le 2\exp(-\tfrac{1}{2}x^2/n), \qquad x > 0.$$

4.9. *Russo's formula.* Let X be a random variable on the finite sample space $\Omega = \{0, 1\}^E$. Show that

$$\frac{d}{dp}\mathbb{P}_p(X) = \sum_{e \in E} \mathbb{P}_p(\delta_e X)$$

where $\delta_e X(\omega) = X(\omega^e) - X(\omega_e)$, and ω^e (respectively, ω_e) is the configuration obtained from ω by replacing $\omega(e)$ by 1 (respectively, 0).

Let A be an increasing event, with indicator function 1_A. An edge e is called *pivotal* for the event A in the configuration ω if $\delta_e I_A(\omega) = 1$. Show that the derivative of $P_p(A)$ equals the mean number of pivotal edges for A. Find a related formula for the second derivative of $P_p(A)$.

What can you show for the third derivative, and so on?

4.10. [89] Show that every increasing subset of the cube $[0,1]^N$ is Lebesgue-measurable.

4.11. Heads turn up with probability p on each of N coin flips. Let A be an increasing event, and suppose there exists a subgroup $\mathcal{A}$ of permutations of $\{1, 2, \dots, N\}$ acting transitively, such that A is $\mathcal{A}$-invariant. Let p_c be the value of p such that $\mathbb{P}_p(A) = \frac{1}{2}$. Show that there exists a constant $c > 0$ such that

$$\mathbb{P}_p(A) \geq 1 - \tfrac{1}{2} N^{-c(p-p_c)}, \qquad p \geq p_c,$$

with a similar inequality for $p \leq p_c$.

4.12. Let μ be a positive measure on $\Omega = \{0,1\}^E$ satisfying the FKG lattice condition. For $p \in (0,1)$, let μ_p be the probability measure given by

$$\mu_p(\omega) = \frac{1}{Z_p}\left(\prod_{e\in E} p^{\omega(e)}(1-p)^{1-\omega(e)}\right)\mu(\omega), \qquad \omega \in \Omega.$$

Let A be an increasing event. Show that

$$\mu_{p_1}(A)(1-\mu_{p_2}(A)) \leq \lambda^{B(p_2-p_1)}, \qquad 0 < p_1 < p_2 < 1,$$

where

$$B = \inf_{p\in(p_1,p_2)} \left\{\frac{c\xi_p}{p(1-p)}\right\}, \qquad \xi_p = \min_{e\in E}\big[\mu_p(\omega(e)=1)\mu_p(\omega(e)=0)\big],$$

and λ satisfies

$$2\max_{e\in E} J_A(e) \leq \lambda, \qquad e \in E,\ p \in (p_1, p_2).$$

5

Further Percolation

The subcritical and supercritical phases of percolation are characterized respectively by the absence and presence of an infinite open cluster. Connection probabilities decay exponentially when $p < p_c$, and there is a unique infinite cluster when $p > p_c$. The power-law singularity at the phase transition is summarized. It is shown that $p_c = \frac{1}{2}$ for bond percolation on the square lattice. The Russo–Seymour–Welsh (RSW) method is described for site percolation on the triangular lattice, and this leads to a statement and proof of Cardy's formula.

5.1 Subcritical phase

In language borrowed from the theory of branching processes, a percolation process is termed *subcritical* if $p < p_c$, and *supercritical* if $p > p_c$.

In the subcritical phase, all open clusters are (almost surely) finite. The chance of a long-range connection is small, and it approaches zero as the distance between the endpoints diverges. The process is considered to be 'disordered', and the probabilities of long-range connectivities tend to zero *exponentially* in the distance. Exponential decay may be proved by elementary means for sufficiently small p, as in the proof of Theorem 3.2, for example. It is quite another matter to prove exponential decay for all $p < p_c$, and this was achieved for percolation by Aizenman and Barsky [6] and Menshikov [164, 165] around 1986. The principal result is the following therem, in which $B(n) = [-n, n]^d$ and $\partial B(n) = B(n) \setminus B(n-1)$.

(5.1) Theorem [6, 164]. *There exists $\psi(p)$, satisfying $\psi(p) > 0$ when $0 < p < p_c$, such that*

(5.2) $$P_p(0 \leftrightarrow \partial B(n)) \le e^{-n\psi(p)}, \qquad n \ge 1.$$

The reader is referred to [95] for a full account of this important theorem. The two proofs of Aizenman–Barsky and Menshikov have some interesting similarities, while differing in fundamental ways. An outline of Menshikov's proof is presented later in this section. The Aizenman–Barsky proof proceeds via an intermediate result, namely the following of Hammersley [114].

(5.3) Theorem [114]. *Suppose that* $\chi(p) = E_p|C| < \infty$. *There exists* $\sigma(p) > 0$ *such that*

$$P_p(0 \leftrightarrow \partial B(n)) \le e^{-n\sigma(p)}, \qquad n \ge 1. \tag{5.4}$$

Seen in the light of Theorem 5.1, one may take the condition $\chi(p) < \infty$ as a characterization of the subcritical phase. It is not difficult to see, using subadditivity, that the limit of $n^{-1} \log P_p(0 \leftrightarrow \partial B(n))$ exists as $n \to \infty$. See [95, Thm 6.10].

Proof. Let $x \in \partial B(n)$, and let $\tau_p(0, x) = P_p(0 \leftrightarrow x)$ be the probability that there exists an open path of $\mathbb{L}^d$ joining the origin to x. Let N_n be the number of vertices $x \in \partial B(n)$ with this property, so that the mean value of N_n is

$$E_p(N_n) = \sum_{x \in \partial B(n)} \tau_p(0, x). \tag{5.5}$$

Note that

$$\begin{aligned} \sum_{n=0}^{\infty} E_p(N_n) &= \sum_{n=0}^{\infty} \sum_{x \in \partial B(n)} \tau_p(0, x) \\ &= \sum_{x \in \mathbb{Z}^d} \tau_p(0, x) \\ &= E_p\big|\{x \in \mathbb{Z}^d : 0 \leftrightarrow x\}\big| = \chi(p). \end{aligned} \tag{5.6}$$

If there exists an open path from the origin to some vertex of $\partial B(m + k)$, then there exists a vertex x in $\partial B(m)$ that is connected by disjoint open paths both to the origin and to a vertex on the surface of the translate $\partial B(k, x) = x + B(k)$, see Figure 5.1. By the BK inequality,

$$\begin{aligned} P_p(0 \leftrightarrow \partial B(m + k)) &\le \sum_{x \in \partial B(m)} P_p(0 \leftrightarrow x) P_p(x \leftrightarrow x + \partial B(k)) \\ &= \sum_{x \in \partial B(m)} \tau_p(0, x) P_p(0 \leftrightarrow \partial B(k)) \end{aligned} \tag{5.7}$$

by translation-invariance. Thus

$$P_p(0 \leftrightarrow \partial B(m + k)) \le E_p(N_m) P_p(0 \leftrightarrow \partial B(k)), \qquad m, k \ge 1. \tag{5.8}$$

The BK inequality makes this calculation simple, Hammersley [114] employed a more elaborate argument.

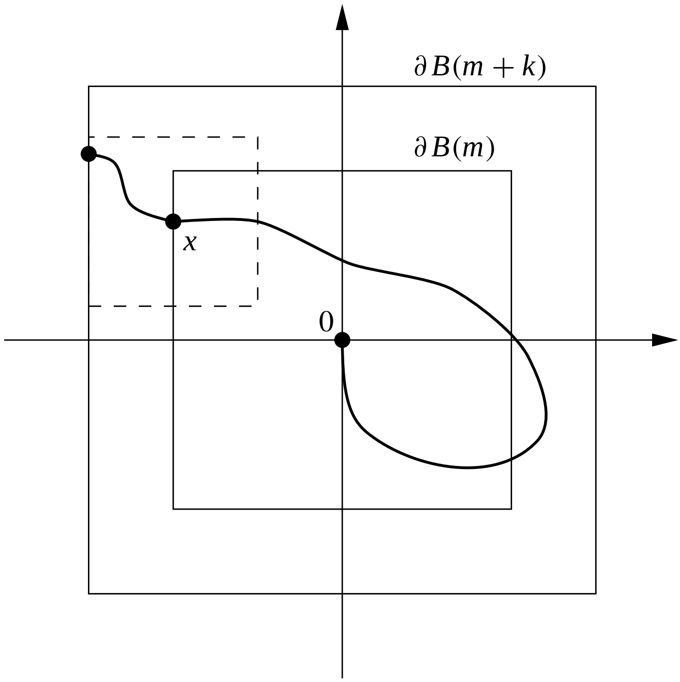

Figure 5.1. The vertex x is joined by disjoint open paths to the origin and to the surface of the translate $B(k, x) = x + B(k)$, indicated by the dashed lines.

Let p be such that $\chi(p) < \infty$, so that $\sum_{m=0}^{\infty} E_p(N_m) < \infty$ from (5.6). Then $E_p(N_m) \to 0$ as $m \to \infty$, and we may choose m such that $\eta = E_p(N_m)$ satisfies $\eta < 1$. Let n be a positive integer and write $n = mr + s$ where r and s are non-negative integers and $0 \le s < m$. Then

$$\begin{aligned} P_p(0 \leftrightarrow \partial B(n)) &\le P_p(0 \leftrightarrow \partial B(mr)) && \text{since } n \ge mr \\ &\le \eta^r && \text{by iteration of (5.8)} \\ &\le \eta^{-1+n/m} && \text{since } n < m(r+1), \end{aligned}$$

which provides an exponentially decaying bound of the form of (5.4), valid for $n \ge m$. It is left as an exercise to extend the inequality to $n < m$. □

Outline proof of Theorem 5.1. The full proof can be found in [94, 95, 165, 210]. Let $S(n)$ be the 'diamond' $S(n) = \{x \in \mathbb{Z}^d : d(0, x) \le n\}$ containing all points within graph-theoretic distance n of the origin, and write $A_n = \{0 \leftrightarrow \partial S(n)\}$. We are concerned with the probabilities $g_p(n) = P_p(A_n)$.

By Russo's formula, Theorem 4.75,

$$g_p'(n) = E_p(N_n) \tag{5.9}$$

where N_n is the number of pivotal edges for A_n, that is, the number of edges e for which $1_A(\omega^e) \ne 1_A(\omega_e)$. By a simple calculation

$$g_p'(n) = \frac{1}{p} E_p(N_n 1_{A_n}) = \frac{1}{p} E_p(N_n \mid A_n) g_p(n), \tag{5.10}$$

which may be integrated to obtain

$$(5.11)\qquad g_\alpha(n) = g_\beta(n)\exp\left(-\int_\alpha^\beta \frac{1}{p}\,E_p(N_n \mid A_n)\,dp\right) \le g_\beta(n)\exp\left(-\int_\alpha^\beta E_p(N_n \mid A_n)\,dp\right),$$

where $0 < \alpha < \beta < 1$. The vast majority of the work in the proof is devoted to showing that $E_p(N_n \mid A_n)$ grows at least linearly in n when $p < p_c$, and the conclusion of the theorem then follows immediately.

The rough argument is as follows. Let $p < p_c$, so that $P_p(A_n) \to 0$ as $n \to \infty$. In calculating $E_p(N_n \mid A_n)$, we are conditioning on an event of diminishing probability, and thus it is feasible that there are many pivotal edges of A_n. This will be proved by bounding (above) the mean distance between consecutive pivotal edges, and then applying a version of Wald's equation. The BK inequality, Theorem 4.14, plays an important role.

Suppose that A_n occurs, and denote by $e_1, e_2, \dots, e_N$ the pivotal edges for A_n, in the order in which they are encountered when building the cluster from the origin. It is easily seen that all open paths from the origin to $\partial S(n)$ traverse every e_j. Furthermore, as illustrated in Figure 5.2, there must exist at least two edge-disjoint paths from the second endpoint of each e_j (in the above ordering) to the first of e_{j+1}.

Let $M = \max\{k : A_k \text{ occurs}\}$, so that

$$P_p(M \ge k) = g_p(k) \to 0 \qquad \text{as } k \to \infty.$$

The key inequality states that

$$(5.12)\qquad P_p(N_n \ge k \mid A_n) \ge \mathbb{P}(M_1 + M_2 + \cdots + M_k \le n - k),$$

where the M_i are independent copies of M. This is proved using the BK inequality, using the above observation concerning disjoint paths between consecutive pivotal edges. The proof is omitted here. By (5.12),

$$(5.13)\qquad P_p(N_n \ge k \mid A_n) \ge P(M_1' + M_2' + \cdots + M_k' \le n),$$

where $M_i' = 1 + \min\{M_i, n\}$. Summing (5.13) over k, we obtain

$$(5.14)\qquad E_p(N_n \mid A_n) \ge \sum_{k=1}^{\infty} P(M_1' + M_2' + \cdots + M_k' \le n) = \sum_{k=1}^{\infty} P_p(K \ge k+1) = E(K) - 1,$$

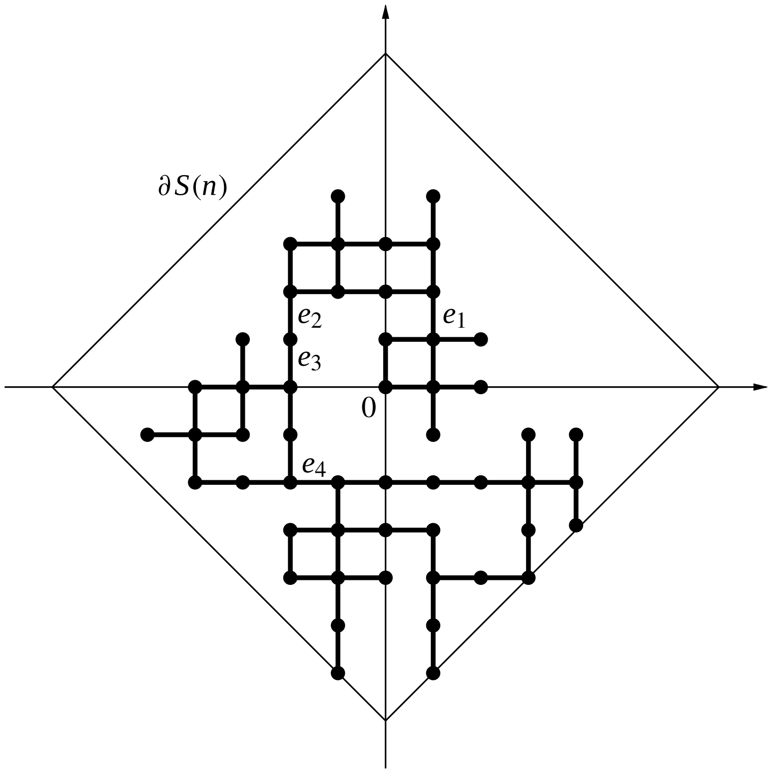

Figure 5.2. Assume that $0 \leftrightarrow \partial S(n)$. For any consecutive pair e_j, e_{j+1} of pivotal edges, taken in the order of traversal from 0 to $\partial S(n)$, there must exist at least two edge-disjoint open paths joining the second vertex of e_j and the first of e_{j+1}.

where $K = \min\{k : M_1' + M_2' + \cdots + M_k' > n\}$. By Wald's equation,

$$n < E(S_K) = E(K)E(M_1'),$$

whence

$$E(K) > \frac{n}{E(M_1')} = \frac{n}{1 + E(\min\{M_1, n\})} = \frac{n}{\sum_{i=0}^{n} g_p(i)}.$$

In summary, this shows that

$$E_p(N_n \mid A_n) \geq \frac{n}{\sum_{i=0}^{n} g_p(i)} - 1, \qquad 0 < p < 1. \tag{5.15}$$

Inequality (5.15) may be fed into (5.10) to obtain a differential inequality for the $g_p(k)$. By a careful analysis of the latter inequality, one obtains that $E_p(N_n \mid A_n)$ grows at least linearly with n whenever p satisfies $0 < p < p_c$. This step is neither short nor easy, but it is conceptually straightforward, and it completes the proof. □

5.2 Supercritical phase

The critical value p_c is the value of p at which the percolation probability $\theta(p)$ becomes strictly positive. It is widely believed that $\theta(p_c) = 0$, and this is perhaps the major conjecture of the subject.

(5.16) Conjecture. *For percolation on $\mathbb{L}^d$ with $d \geq 2$, we have that $\theta(p_c) = 0$.*

It is known that $\theta(p_c) = 0$ when either $d = 2$ (by results of [121], see Theorem 5.33) or $d \geq 19$ (by the lace expansion of [118, 119]). The claim is believed to be canonical of percolation models on all lattices and in all dimensions.

Suppose now that $p > p_c$, so that $\theta(p) > 0$. What can be said about the number N of infinite open clusters? Since the event $\{N \geq 1\}$ is translation-invariant, it is trivial under the product measure P_p. However, $P_p(N \geq 1) \geq \theta(p) > 0$, whence

$$P_p(N \geq 1) = 1, \qquad p > p_c.$$

We shall see in the forthcoming Theorem 5.22 that $P_p(N = 1) = 1$ whenever $\theta(p) > 0$, which is to say that there exists a unique infinite open cluster throughout the supercritical phase.

A supercritical percolation process in *two* dimensions may be studied in either of two ways. The first of these is by duality. Consider bond percolation on $\mathbb{L}^2$ with density p. The dual process (as in the proof of the upper bound of Theorem 3.2) is bond percolation with density $1 - p$. We shall see in Theorem 5.33 that the self-dual point $p = \frac{1}{2}$ is also the critical point. Thus, the dual of a supercritical process is subcritical, and this enables a study of supercritical percolation on $\mathbb{L}^2$. A similar argument is valid for certain other lattices, although it is clear that the square lattice is special in that it is a self-dual graph.

While duality is the technique for studying supercritical percolation in two dimensions, the process may also be studied by the block argument that follows. The block method was devised expressly for three and more dimensions in the hope that, amongst other things, it would imply the claim of Conjecture 5.16. Block arguments are a work-horse of the theory of general interacting systems.

We assume henceforth that $d \geq 3$ and that p is such that $\theta(p) > 0$; under this hypothesis, we wish to gain some control of the (a.s.) unique open cluster. The main result is the following, of which an outline proof is included later in the section. Let $A \subseteq \mathbb{Z}^d$, and write $p_c(A)$ for the critical probability of bond percolation on the subgraph of $\mathbb{Z}^d$ induced by A. Thus, for example, $p_c = p_c(\mathbb{Z}^d)$. Recall that $B(k) = [-k, k]^d$.

(5.17) Theorem [103]. *Let $d \geq 3$. If F is an infinite connected subset of $\mathbb{Z}^d$ with $p_c(F) < 1$, then for each $\eta > 0$ there exists an integer k such that*

$$p_c(2kF + B(k)) \leq p_c + \eta.$$

That is, for any set F sufficiently large that $p_c(F) < 1$, one may 'fatten' F to a set having critical probability as close to p_c as required. One particular application of this theorem is to the limit of slab critical probabilities, and we elaborate on this next.

Many results have been proved for subcritical percolation under the 'finite susceptibility' hypothesis that $\chi(p) < \infty$. The validity of this hypothesis for $p < p_c$ is implied by Theorem 5.1. Similarly, several important results for supercritical percolation have been proved under the hypothesis that 'percolation occurs in slabs'. The two-dimensional slab F_k of thickness $2k$ is the set

$$F_k = \mathbb{Z}^2 \times [-k, k]^{d-2} = \left(\mathbb{Z}^2 \times \{0\}^{d-2}\right) + B(k),$$

with critical probability $p_c(F_k)$. Since $F_k \subseteq F_{k+1} \subseteq \mathbb{Z}^d$. the decreasing limit $p_c(F) = \lim_{k\to\infty} p_c(F_k)$ exists and satisfies $p_c(F) \geq p_c$. The hypothesis of 'percolation in slabs' is that $p > p_c(F)$. By Theorem 5.17,

$$\lim_{k\to\infty} p_c(F_k) = p_c, \tag{5.18}$$

On of the best examples of the use of 'slab percolation' is the following estimate of the extent of a finite open cluster.

(5.19) Theorem [60]. *The limit*

$$\sigma(p) = \lim_{n\to\infty} \left\{ -\frac{1}{n} \log P_p(0 \leftrightarrow \partial B(n),\ |C| < \infty) \right\}$$

exists. Furthermore $\sigma(p) > 0$ *if* $p > p_c$.

This theorem asserts the exponential decay of a 'truncated' connectivity function when $d \geq 3$. A similar result may be proved by duality for $d = 2$.

We turn briefly to a discussion of the so-called 'Wulff crystal'. Much attention has been paid to the sizes and shapes of clusters formed in models of statistical mechanics. When a cluster C is infinite with a strictly positive probability, but is constrained to have some large *finite* size n, then C is said to form a large 'droplet'. The asymptotic shape of such a droplet as $n \to \infty$ is prescribed in general terms by the theory of the so-called Wulff crystal, see the original paper [216] of Wulff. Specializing to percolation, we ask for properties of the open cluster C at the origin, conditioned on the event $\{|C| = n\}$.

The study of the Wulff crystal is bound up with the law of the volume of a finite cluster. This has a tail that is 'quenched exponential':

$$P_p(|C| = n) \approx \exp(-\rho n^{(d-1)/d}), \tag{5.20}$$

where $\rho = \rho(p) \in (0, \infty)$ for $p > p_c$, and $\approx$ is to be interpreted in terms of exponential asymptotics. The explanation for the curious exponent is as follows. The 'most economic' way to create a large finite cluster is to find a region R containing a connected component D of size n, satisfying $D \leftrightarrow \infty$, and then

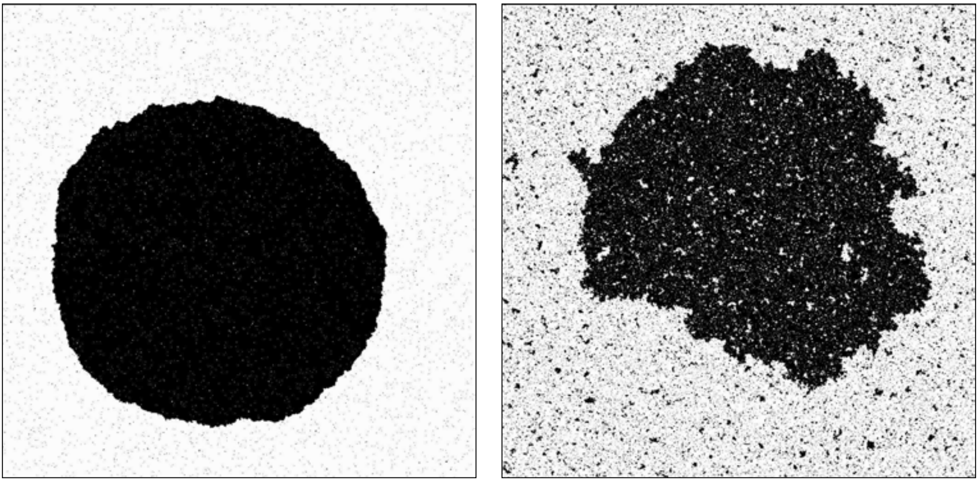

Figure 5.3. Images of the Wulff crystal for the two-dimensional Ising model at two distinct temperatures, produced by simulation in time. The simulations were for finite time, and the images are therefore only approximations to the true crystals. The pictures are 1024 pixels square, and the inverse-temperatures are $\beta = \frac{4}{3}, \frac{10}{11}$. The corresponding random-cluster models have $q = 2$ and $p = 1 - e^{-4/3}, 1 - e^{-10/11}$.

to cut all connections leaving R. Since $p > p_c$, such regions R exist with $|R|$ (respectively, $|\partial R|$) having order n (respectively, $n^{(d-1)/d}$), and the 'cost' of the construction is exponential in $|\partial R|$.

The above argument yields a lower bound for $P_p(|C| = n)$ of the quenched-exponential type, but considerably more work is required to show the exact asymptotic of (5.20), and indeed one obtains more. The (conditional) shape of $Cn^{-1/d}$ converges as $n \to \infty$ to the solution of a certain variational problem, and the asymptotic region is termed the 'Wulff crystal' for the model. This is not too hard to make rigorous when $d = 2$, since the external boundary of C is then a closed curve. Serious technical difficulties arise when pursuing this programme when $d \geq 3$. See [55] for an account and a bibliography.

Outline proof of Theorem 5.19. The existence of the limit is an exercise in subadditivity of a standard type, although with some complications in this case (see [59, 95]). We sketch here a proof of the important estimate $\sigma(p) > 0$.

Let S_k be the $(d-1)$-dimensional slab

$$S_k = [0, k] \times \mathbb{Z}^{d-1}.$$

Since $p > p_c$, we have by Theorem 5.17 that $p > p_c(S_k)$ for some k, and we choose k accordingly. Let H_n be the hyperplane of vertices x of $\mathbb{L}^d$ with $x_1 = n$. It suffices to prove that

$$P_p(0 \leftrightarrow H_n,\ |C| < \infty) \leq e^{-\gamma n} \tag{5.21}$$

for some $\gamma = \gamma(p) > 0$. Define the slabs

$$T_i = \{x \in \mathbb{Z}^d : (i-1)k \leq x_1 < ik\}, \quad 1 \leq i < \lfloor n/k \rfloor.$$

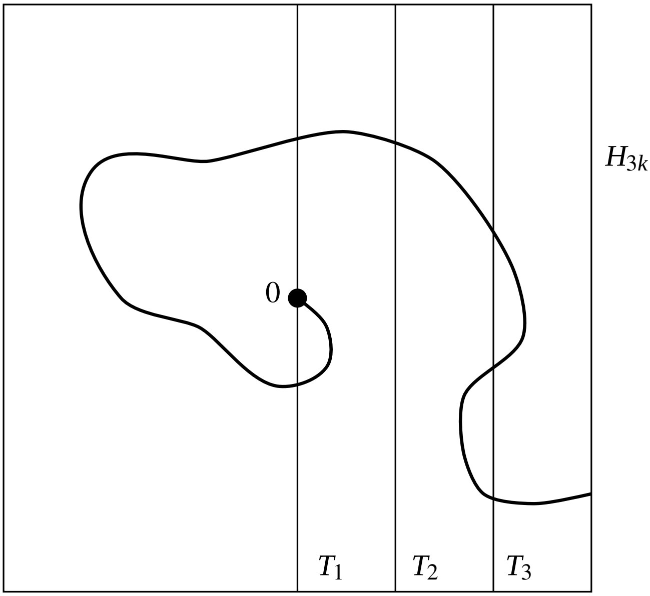

Figure 5.4. All paths from the origin to H_{3k} traverse the regions T_i, $i = 1, 2, 3$.

Any path from 0 to H_n traverses each T_i. Since $p > p_c(S_k)$, each slab contains (a.s.) an infinite open cluster. See Figure 5.4. If $0 \leftrightarrow H_n$ and $|C| < \infty$, then all paths from 0 to H_n must evade all such clusters. There are $\lfloor n/k \rfloor$ slabs to traverse, and a price is paid for each. Modulo a touch of rigour, this implies that

$$P_p(0 \leftrightarrow H_n,\ |C| < \infty) \le [1 - \theta_k(p)]^{\lfloor n/k \rfloor}$$

where

$$\theta_k(p) = P_p(0 \leftrightarrow \infty \text{ in } S_k) > 0.$$

The inequality $\sigma(p) > 0$ is proved. □

Outline proof of Theorem 5.17. The full proof can be found in [95, 103]. For simplicity, we shall take $F = \mathbb{Z}^2 \times \{0\}^{d-2}$, so that $2kF + B(k) = \mathbb{Z}^2 \times [-k, k]^{d-2}$. There are two main steps in the proof. In the first, we show the existence of long finite paths. In the second, we show how to take such finite paths and build an infinite cluster in a slab. The principal parts of the first step are as follows. Let p be such that $\theta(p) > 0$.

1. Let $\epsilon > 0$. Since $\theta(p) > 0$, there exists m such that

$$P_p(B(m) \leftrightarrow \infty) > 1 - \epsilon.$$

[*This holds since there exists, almost surely, an infinite open cluster*.]

2. Let $n \ge 2m$, say, and let $k \ge 1$. We may choose n sufficiently large that, with probability at least $1 - 2\epsilon$, $B(m)$ is joined to at least k points in $\partial B(n)$.

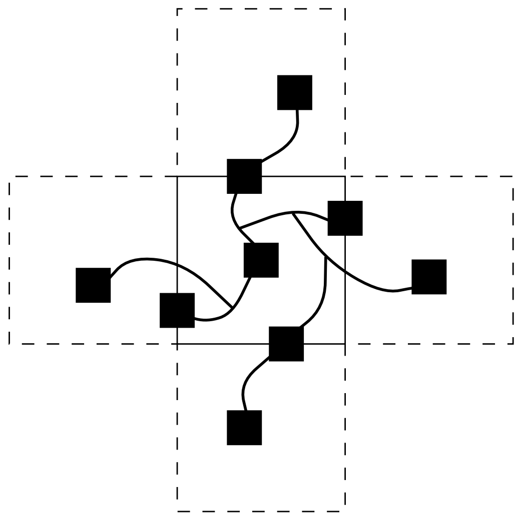

Figure 5.5. An illustration of the event that the block centred at the origin is open. Each black square is a seed.

[*If, for some k, this fails for unbounded n, then there exists N such that* $B(m) \nleftrightarrow \mathbb{Z}^d \setminus B(N)$.]

3. By choosing k sufficiently large, we may ensure that, with probability at least $1-3\epsilon$, $B(m)$ is joined to some point of $\partial B(n)$, which is itself connected to a copy of $B(m)$, lying 'on' the surface $\partial B(n)$ and every edge of which is open. [*We may choose k sufficiently large that there are many non-overlapping copies of* $B(m)$ *in the correct positions, indeed sufficiently many that, with high probability, one is totally open.*]
4. The open copy of $B(m)$, constructed above, may be used as a 'seed' for iterating the above construction. When doing this, we shall need some control over where the seed is placed. It may be shown that every face of $\partial B(n)$ contains (with large probability) a point adjacent to some seed, and indeed many such points. See Figure 5.5. [*There is sufficient symmetry to deduce this by the FKG inequality.*]

Above is the scheme for constructing long finite paths, and we turn to the second step.

5. This construction is now iterated. At each stage there is a certain (small) probability of failure. In order that there be a strictly positive probability of an infinite sequence of successes, we iterate 'in two independent directions'. With care, one may show that the construction dominates a certain supercritical site percolation process on $\mathbb{L}^2$.
6. We wish to deduce that an infinite sequence of successes entails an infinite open path of $\mathbb{L}^d$ within the corresponding slab. There are two difficulties with this. First, since we do not have total control of the positions of the seeds, the actual path in $\mathbb{L}^d$ may leave every slab. This may be overcome by a process of 'steering', in which, at each stage, we choose a seed in such

a position as to compensate for any earlier deviation in space.

7. A greater problem is that, in iterating the construction, we carry with us a mixture of 'positive' and 'negative' information (of the form that 'certain paths exist' and 'others do not'). In combining events we cannot use the FKG inequality. The practical difficulty is that, although we may have an infinite sequence of successes, there will generally be breaks in any corresponding open route to ∞. This is overcome by sprinkling down a few more open edges, that is, by working at edge-density $p+\delta$ where $\delta > 0$, rather than at density p.

In conclusion, we find that, if $\theta(p) > 0$ and $\delta > 0$, then there exists, with large probability, an infinite $(p+\delta)$-open path in a slice of the form $T_k = \mathbb{Z}^2 \times [-k, k]^{d-2}$ for sufficiently large k. The claim of the theorem follows.

There are many details to be considered in carrying out the above programme, and these are omitted here. □

5.3 Uniqueness of the infinite cluster

The principal result of this section is the following: for any value of p for which $\theta(p) > 0$, there exists (a.s.) a unique infinite open cluster. Let $N = N(\omega)$ be the number of infinite open clusters.

(5.22) Theorem [12]. *If $\theta(p) > 0$, then $P_p(N = 1) = 1$.*

A similar conclusion holds for more general probability measures. The two principal ingredients of the generalization are the translation-invariance of the measure, and the so-called 'finite-energy property', that states that, conditional on the states of all edges except e, say, the state of e is 0 (respectively, 1) with a strictly positive (conditional) probability.

Proof. We follow [50]. The claim is trivial if $p = 0, 1$, and we assume henceforth that $0 < p < 1$. Let $S = S(n)$ be the 'diamond' $S(n) = \{x \in \mathbb{Z}^d : d(0, x) \le n\}$, and let $\mathbb{E}_S$ be the set of edges of $\mathbb{L}^d$ joining pairs of vertices in S. We write $N_S(0)$ (respectively $N_S(1)$) for the total number of infinite open clusters when all edges in $\mathbb{E}_S$ are declared to be closed (respectively open). Finally, M_S denotes the number of infinite open clusters that intersect S.

The sample space $\Omega = \{0, 1\}^{\mathbb{E}^d}$ is a product space with a natural family of translations, and P_p is a product measure on Ω. Since N is a translation-invariant function on Ω, it is almost surely constant, which is to say that

(5.23) there exists $k = k(p) \in \{0, 1, 2, \dots\} \cup \{\infty\}$ such that $P_p(N = k) = 1$.

Next we show that the k in (5.23) necessarily satisfies $k \in \{0, 1, \infty\}$. Suppose that (5.23) holds with $k < \infty$. Since every configuration on $\mathbb{E}_S$ has a strictly positive probability, it follows by the almost sure constantness of N that

$$P_p(N_S(0) = N_S(1) = k) = 1.$$

Now $N_S(0) = N_S(1)$ if and only if S intersects at most one infinite open cluster (this is where we use the assumption that $k < \infty$), and therefore

$$P_p(M_S \geq 2) = 0.$$

Clearly, M_S is non-decreasing in $S = S(n)$, and $M_{S(n)} \to N$ as $n \to \infty$. Therefore,

$$0 = P_p(M_{S(n)} \geq 2) \to P_p(N \geq 2), \tag{5.24}$$

which is to say that $k \leq 1$.

It remains to rule out the case $k = \infty$. Suppose that $k = \infty$. We will derive a contradiction by using a geometrical argument. We call a vertex x a *trifurcation* if:

(a) x lies in an infinite open cluster,
(b) there exist exactly three open edges incident to x, and
(c) the deletion of x and its three incident open edges splits this infinite cluster into exactly three disjoint infinite clusters and no finite clusters;

Let T_x be the event that x is a trifurcation. By translation-invariance, $P_p(T_x)$ is constant for all x, and therefore

$$\frac{1}{|S(n)|} E_p\biggl(\sum_{x \in S(n)} 1_{T_x}\biggr) = P_p(T_0). \tag{5.25}$$

It will be useful to know that the quantity $P_p(T_0)$ is strictly positive, and it is here that we use the assumed infinity of infinite clusters. Let $M_S(0)$ be the number of infinite open clusters that intersect S when all edges of $\mathbb{E}_S$ are declared closed. Since $M_S(0) \geq M_S$, by the remarks around (5.24),

$$P_p(M_{S(n)}(0) \geq 3) \geq P_p(M_{S(n)} \geq 3) \to P_p(N \geq 3) = 1 \qquad \text{as } n \to \infty.$$

Therefore, there exists m such that

$$P_p(M_{S(m)}(0) \geq 3) \geq \tfrac{1}{2},$$

and we set $S = S(m)$. Note that:

(a) the event $\{M_S(0) \geq 3\}$ is independent of the states of edges in $\mathbb{E}_S$,
(b) if the event $\{M_S(0) \geq 3\}$ occurs, there exist $x, y, z \in \partial S$ lying in distinct infinite open clusters of $\mathbb{E}^d \setminus \mathbb{E}_S$.

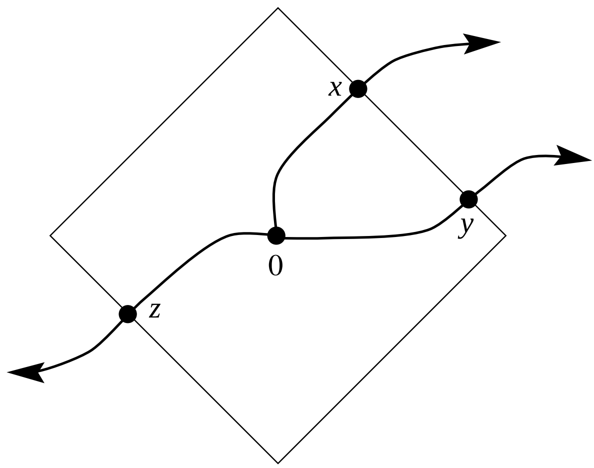

Figure 5.6. Take a diamond S that intersects at least three distinct infinite open clusters, and then alter the configuration inside S in order to create a configuration in which 0 is a trifurcation.

Let $\omega \in \{M_S(0) \geq 3\}$, and pick $x = x(\omega)$, $y = y(\omega)$, $z = z(\omega)$ according to (b). If there is more than one possible such triple, we pick such a triple according to some predetermined rule. It is a minor geometrical exercise (see Figure 5.6) to verify that there exist in $\mathbb{E}_S$ three paths joining the origin to (respectively) x, y, and z, and that these paths may be chosen in such a way that:

(i) the origin is the unique vertex common to any two of them, and

(ii) each touches exactly one vertex lying in ∂S.

Let $J_{x,y,z}$ be the event that all the edges in these paths are open, and that all other edges in $\mathbb{E}_S$ are closed.

Since S is finite,

$$P_p(J_{x,y,z} \mid M_S(0) \geq 3) \geq \left[\min\{p, 1-p\}\right]^R > 0,$$

where $R = |\mathbb{E}_S|$. Now,

$$\begin{aligned} P_p(0 \text{ is a trifurcation}) &\geq P_p(J_{x,y,z} \mid M_S(0) \geq 3) P_p(M_S(0) \geq 3) \\ &\geq \tfrac{1}{2}\left[\min\{p, 1-p\}\right]^R > 0, \end{aligned}$$

which is to say that $P_p(T_0) > 0$ in (5.25).

It follows from (5.25) that the mean number of trifurcations inside $S = S(n)$ grows in the manner of $|S|$ as $n \to \infty$. On the other hand, we shall see next that the number of trifurcations inside S can be no larger than the size of the boundary of S, and this provides the necessary contradiction. This final step must be performed properly (see [50, 95]), but the following rough argument is appealing and may be made rigorous. Select a trifurcation (t_1, say) of S, and choose some vertex $y_1 \in \partial S$ such that $t_1 \leftrightarrow y_1$ in S. We now select a new trifurcation $t_2 \in S$. It may be seen, using the definition of the term 'trifurcation', that there exists $y_2 \in \partial S$ such that $y_1 \neq y_2$ and $t_2 \leftrightarrow y_2$ in S. We continue similarly, at each stage picking a new trifurcation $t_k \in S$ and a new vertex $y_k \in \partial S$. If there are τ trifurcations in S, then we obtain τ distinct vertices y_k of ∂S. Therefore $|\partial S| \geq \tau$. However, by the remarks above, $E_p(\tau)$ is comparable to S. This is a contradiction for large n, since $|\partial S|$ grows in the manner of n^{d-1} and $|S|$ grows in the manner of n^d. □

5.4 Phase transition

Macroscopic functions, such as the percolation probability $\theta(p) = P_p(|C| = \infty)$ and the mean cluster size $\chi(p) = E_p|C|$, have singularities at $p = p_c$, and there is overwhelming evidence that these are of 'power law' type. A great deal of effort has been invested by physicists and mathematicians in understanding the nature of the percolation phase-transition. The picture is now fairly clear when $d = 2$, owing to the very significant progress in recent years in relating critical percolation to the stochastic (Schramm–)Löwner curve SLE_6. There remain however substantial difficulties to be overcome before this chapter of percolation theory can be declared written, even when $d = 2$. The case of large d (currently, $d \geq 19$) is also well understood, through work based on the so-called 'lace expansion'. Most problems remain open in the obvious case $d = 3$, and ambitious and brave students are directed thus, with caution.

The nature of the percolation singularity is supposed to be canonical, in that it is expected to have certain general features in common with phase transitions of other models of statistical mechanics. These features are sometimes referred to as 'scaling theory' and they relate to 'critical exponents'. There are two sets of critical exponents, arising firstly in the limit as $p \to p_c$, and secondly in the limit over increasing distances when $p = p_c$. We summarize the notation in Table 5.8.

The asymptotic relation $\approx$ should be interpreted loosely (perhaps via logarithmic asymptotics[1]). The radius of C is defined by

$$\operatorname{rad}(C) = \max\{n : 0 \leftrightarrow \partial[-n, n]^d\}.$$

The limit as $p \to p_c$ should be interpreted in a manner appropriate for the function in question (for example, as $p \downarrow p_c$ for $\theta(p)$, but as $p \to p_c$ for $\kappa(p)$).

There are eight critical exponents listed in Table 5.8, denoted α, β, γ, δ, ν, η, ρ, Δ, but there is no general proof of the existence of any of these exponents for arbitrary d. In general, the eight critical exponents may be defined for phase transitions in a quite large family of physical systems. However, it is not believed

[1]We say that $f(x)$ is logarithmically asymptotic to $g(x)$ as $x \to 0$ (respectively, $x \to \infty$) if $\log f(x)/\log g(x) \to 1$. This is often written as $f(x) \approx g(x)$.

Function		Behaviour	Exponent
percolation probability	$\theta(p) = P_p(\lvert C\rvert = \infty)$	$\theta(p) \approx (p - p_c)^\beta$	β
truncated mean cluster size	$\chi^f(p) = E_p(\lvert C\rvert; \lvert C\rvert < \infty)$	$\chi^f(p) \approx \lvert p - p_c\rvert^{-\gamma}$	γ
number of clusters per vertex	$\kappa(p) = E_p(\lvert C\rvert^{-1})$	$\kappa'''(p) \approx \lvert p - p_c\rvert^{-1-\alpha}$	α
cluster moments	$\chi_k^f(p) = E_p(\lvert C\rvert^k; \lvert C\rvert < \infty)$	$\dfrac{\chi_{k+1}^f(p)}{\chi_k^f(p)} \approx \lvert p - p_c\rvert^{-\Delta},\ k \ge 1$	Δ
correlation length	$\xi(p)$	$\xi(p) \approx \lvert p - p_c\rvert^{-\nu}$	ν
cluster volume		$P_{p_c}(\lvert C\rvert = n) \approx n^{-1-1/\delta}$	δ
cluster radius		$P_{p_c}\big(\mathrm{rad}(C) = n\big) \approx n^{-1-1/\rho}$	ρ
connectivity function		$P_{p_c}(0 \leftrightarrow x) \approx \lVert x\rVert^{2-d-\eta}$	η

Table 5.8. Eight functions and their critical exponents.

that they are independent variables, but rather that they satisfy the *scaling relations*:

$$\begin{aligned} 2 - \alpha &= \gamma + 2\beta = \beta(\delta + 1),\\ \Delta &= \delta\beta,\\ \gamma &= \nu(2 - \eta), \end{aligned}$$

and, when d is not too large, the *hyperscaling relations*:

$$\begin{aligned} d\rho &= \delta + 1,\\ 2 - \alpha &= d\nu. \end{aligned}$$

The *upper critical dimension* is the largest value d_c such that the hyperscaling relations hold for $d \le d_c$. It is believed that $d_c = 6$ for percolation. There is no general proof of the validity of the scaling and hyperscaling relations, although quite a lot is known when $d = 2$ and for large d.

In the context of percolation, there is an analytical rationale behind the scaling relations, namely the 'scaling hypotheses' that

$$\begin{aligned} P_p(\lvert C\rvert = n) &\sim n^{-\sigma} f\big(n/\xi(p)^\tau\big)\\ P_p(0 \leftrightarrow x,\ \lvert C\rvert < \infty) &\sim \lVert x\rVert^{2-d-\eta} g\big(\lVert x\rVert/\xi(p)\big) \end{aligned}$$

in the double limit as $p \to p_c$, $n \to \infty$, and for some constants σ, τ, η and functions f, g. Playing loose with rigorous mathematics, the scaling relations may be derived from these hypotheses. Similarly, the hyperscaling relations may be shown to be not too unreasonable, at least when d is not too large. For further discussion, see [95].

We note some further points.

Universality. It is believed that the numerical values of critical exponents depend only on the value of d, and are independent of the particular percolation model.

Two dimensions. When $d = 2$, perhaps

$$\alpha = -\tfrac{2}{3},\ \beta = \tfrac{5}{36},\ \gamma = \tfrac{43}{18},\ \delta = \tfrac{91}{5},\ldots$$

See (5.41).

Large dimension. When d is sufficiently large (actually, $d \ge d_c$) it is believed that the critical exponents are the same as those for percolation on a tree (the 'mean-field model'), namely $\delta = 2$, $\gamma = 1$, $\nu = \frac{1}{2}$, $\rho = \frac{1}{2}$, and so on (the other exponents are found to satisfy the scaling relations). Using the first hyperscaling relation, this supports the contention that $d_c = 6$. Such statements are known to hold for $d \ge 19$, see [118, 119] and the remarks later in this section.

Open challenges include to prove:

– the existence of critical exponents,
– universality,
– the scaling relations,
– the conjectured values when $d = 2$,
– the conjectured values when $d \ge 6$.

Progress towards these goals has been positive. For sufficiently large d, exact values are known for many exponents, namely the values from percolation on a regular tree. There has been remarkable progress in recent years when $d = 2$, inspired largely by work of Schramm [189], enacted by Smirnov [196], and confirmed by the programme pursued by Lawler, Schramm, and Werner to understand SLE curves. See Section 5.6.

We close this section with some further remarks on the case of large d. The expression 'mean-field' permits several interpretations depending on context. A narrow interpretation of the term 'mean-field theory' for percolation involves trees rather than lattices. For percolation on a regular tree, it is quite easy to perform exact calculations of many quantities, including the numerical values of critical exponents. That is, $\delta = 2$, $\gamma = 1$, $\nu = \frac{1}{2}$, $\rho = \frac{1}{2}$, and other exponents are given according to the scaling relations, see [95].

Turning to percolation on $\mathbb{L}^d$, it is known as remarked above that the critical exponents agree with those of a regular tree when d is sufficiently large. In fact, this is believed to hold if and only if $d \ge 6$, but progress so far assumes that $d \ge 19$. In the following theorem, we write $f(x) \simeq g(x)$ if there exist positive constants c_1, c_2 such that $c_1 f(x) \le g(x) \le c_2 f(x)$ for all x close to a limiting value.

(5.26) Theorem [119]. *For $d \ge 19$,*

$$\begin{aligned} \theta(p) &\simeq (p - p_c)^1 && \text{as } p \downarrow p_c, \\ \chi(p) &\simeq (p_c - p)^{-1} && \text{as } p \uparrow p_c, \\ \xi(p) &\simeq (p_c - p)^{-\frac{1}{2}} && \text{as } p \uparrow p_c, \\ \frac{\chi^{f}_{k+1}(p)}{\chi^{f}_{k}(p)} &\simeq (p_c - p)^{-2} && \text{as } p \uparrow p_c, \text{ for } k \ge 1. \end{aligned}$$

Note the strong form of the asymptotic relation $\simeq$, and the identification of the critical exponents β, γ, Δ, ν. The proof of Theorem 5.26 centres on a property known as the 'triangle condition'. Define

(5.27)
$$T(p) = \sum_{x,y \in \mathbb{Z}^d} P_p(0 \leftrightarrow x) P_p(x \leftrightarrow y) P_p(y \leftrightarrow 0),$$

and consider the *triangle condition*:

$$T(p_c) < \infty.$$

The triangle condition was introduced by Aizenman and Newman [15], who showed that it implied that $\chi(p) \simeq (p_c - p)^{-1}$ as $p \uparrow p_c$. Subsequently other authors showed that the triangle condition implied similar asymptotics for other quantities. It was Takashi Hara and Gordon Slade [118] who verified the triangle condition for large d, exploiting a technique known as the 'lace expansion'.

5.5 Open paths in annuli

There is a very useful technique for building open paths with certain geometry in two dimensions. It leads to a proof that the chance of an open circuit within an annulus $[-3n, 3n]^2 \setminus [-n, n]^2$ is at least $f(\delta)$, where δ is the chance of an open crossing of the square $[-n, n]^2$, and f is a strictly positive function. This result was useful in some of the original proofs concerning the critical probability of bond percolation on $\mathbb{L}^2$ (see [95, Sect. 11.7]), and has re-emerged more recently as central to estimates that permit the proof of the Cardy formula and conformal invariance. It is commonly named after Russo [186] and Seymour–Welsh [195]. The RSW lemma will be stated and proved in this section, and utilized in the next three. Since our application in the Section 5.7 will be to site percolation on the triangular lattice, we shall phrase the RSW lemma in that context. It is left to the reader to adapt and develop the arguments of this section for bond percolation on

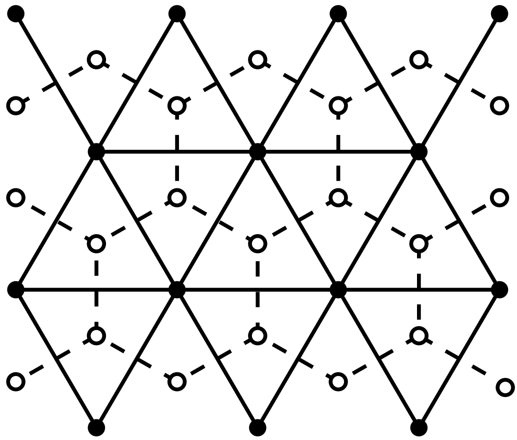

Figure 5.9. The triangular lattice $\mathbb{T}$ and the (dual) hexagonal lattice $\mathbb{H}$.

the square lattice (see Exercise 5.5). The triangular lattice $\mathbb{T}$ is drawn in Figure 5.9, together with its dual hexagonal lattice $\mathbb{H}$.

RSW theory is presented in [95, Sect. 11.7] for the square lattice $\mathbb{L}^2$ and general bond-density p. We could follow the same route here for the triangular lattice, but for the sake of variation (and with an eye to the applications in Section 5.7) we shall restrict ourselves to the case $p = \frac{1}{2}$ and shall give a shortened proof due to Stanislav Smirnov. The more conventional approach may be found in [211], see also [210], and [42] for a variant on the square lattice. Thus, in this section we restrict ourselves to site percolation on $\mathbb{T}$ with density $\frac{1}{2}$. Each site of $\mathbb{T}$ is coloured *black* with probability $\frac{1}{2}$, and *white* otherwise, and the relevant probability measure is denoted as P.

The triangular lattice is embedded in $\mathbb{R}^2$ with vertex-set $\{m\mathbf{i}+n\mathbf{j} : (m, n) \in \mathbb{Z}^2\}$ where $\mathbf{i} = (1, 0)$ and $\mathbf{j} = \frac{1}{2}(1, \sqrt{3})$. Write $R_{a,b}$ for the subgraph induced by vertices in the rectangle $[0, a] \times [0, b]$, and we shall restrict ourselves always to integers a and integer multiples b of $\frac{1}{2}\sqrt{3}$. Let $H_{a,b}$ be the event that there exists a black path that traverses $R_{a,b}$ from its left side to its right side. The 'engine room' of the RSW method is the following lemma.

(5.28) Lemma. $P(H_{2a,b}) \geq \frac{1}{4}P(H_{a,b})^2$.

By iteration,

$$P(H_{2^k a,b}) \geq 4\left[\tfrac{1}{4}P(H_{a,b})\right]^{2^k}, \qquad k \geq 0. \tag{5.29}$$

As 'input' to this inequality, we prove the following.

(5.30) Lemma. *We have that* $P(H_{a,a\sqrt{3}}) \geq \frac{1}{2}$.

Let Λ_m be the set of vertices in $\mathbb{T}$ at graph-theoretic distance m or less from the origin 0, and define the annulus $A_n = \Lambda_{3n} \setminus \Lambda_{n-1}$. Let O_n be the event that A_n contains a black circuit C such that 0 lies in the bounded component of $\mathbb{R}^2 \setminus C$.

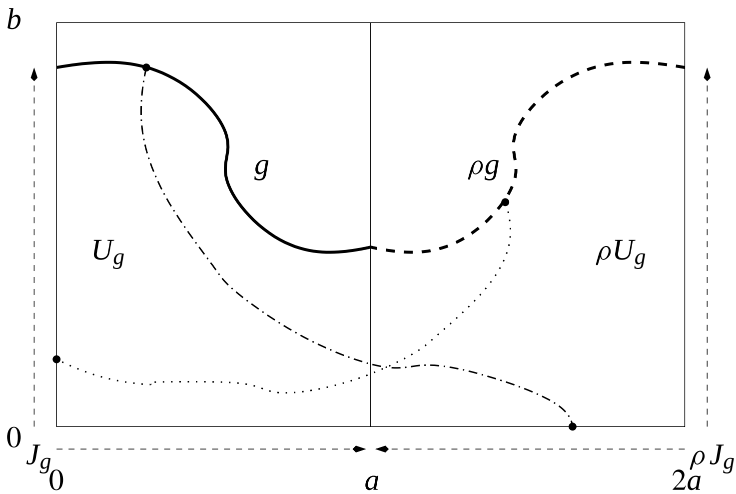

Figure 5.10. The crossing g and its reflection ρg in the box $R_{2a,b}$. The events B_g and $W_{\rho g}$ are illustrated by the two lower paths, and exactly one of these events occurs.

(5.31) Theorem (RSW). *There exists $\sigma > 0$ such that $P(O_n) > \sigma$ for all $n \geq 1$.*

Proof of Lemma 5.28. We follow an unpublished argument of Stanislav Smirnov[2]. Let g be a path that traverses $R_{a,b}$ from left to right. Let ρ denote reflection in the line $x = a$, so that ρg connects the left and right sides of $[a, 2a] \times [0, b]$. See Figure 5.10. Assume for the moment that g does not intersect the x-axis, and let U_g be the connected subgraph of $R_{a,b}$ lying on or 'beneath' g. Let J_g (respectively, J_b) be the part of the boundary ∂U_g (respectively, $R_{a,b}$) lying on either the x-axis or y-axis, and let ρJ_g (respectively, ρJ_b) be its reflection.

Let B_g be the event that there exists a black path of $U_g \cup \rho U_g$ joining some vertex of g to some vertex of ρJ_g. If B_g does not occur, then the event $W_{\rho g}$, that there exists a white path of $U_g \cup \rho U_g$ from ρg to J_g, must occur. The events B_g, $W_{\rho g}$ are mutually exclusive with the same probability, and therefore

$$P(B_g) = P(W_{\rho g}) = \tfrac{1}{2}. \tag{5.32}$$

The same relation holds if g touches the x-axis, with J_g suitably adapted.

Let L be the left side of $R_{2a,b}$ and R its right side. By the FKG inequality,

$$\begin{aligned} P(H_{2a,b}) &\geq P(\mathrm{L} \leftrightarrow \rho J_b,\ \mathrm{R} \leftrightarrow J_b) \\ &\geq P(\mathrm{L} \leftrightarrow \rho J_b) P(\mathrm{R} \leftrightarrow J_b) = P(\mathrm{L} \leftrightarrow \rho J_b)^2, \end{aligned}$$

where $\leftrightarrow$ denotes connection by a black path.

Let γ be the 'highest' black path from the left to the right sides of $R_{a,b}$, if such a path exists. Conditional on the event $\{\gamma = g\}$, the states of sites beneath g are

[2] See also [209].

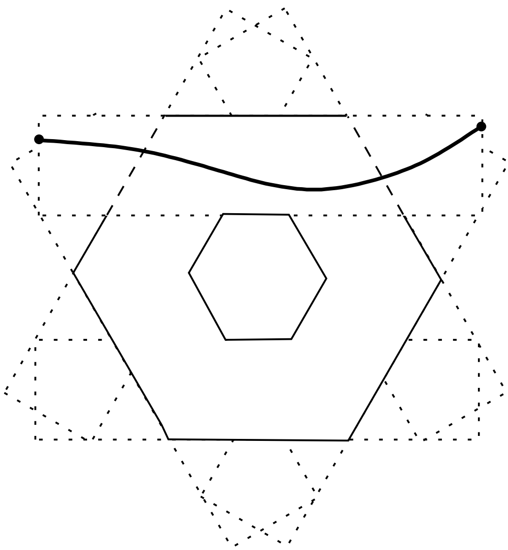

Figure 5.11. If each of six long rectangles are traversed in the long direction by black paths, then the intersection of these paths contains a black cycle within the annulus A_n.

independent Bernoulli variables, whence, in particular, the events B_g and $\{\gamma = g\}$ are independent. By (5.32),

$$P(\mathrm{L} \leftrightarrow \rho J_b) \geq \sum_g P(\gamma = g,\ B_g) = \sum_g P(B_g)P(\gamma = g)$$
$$= \tfrac{1}{2}\sum_g P(\gamma = g) = \tfrac{1}{2}P(H_{a,b}),$$

and the lemma is proved. □

Proof of Lemma 5.30. This is similar to the argument leading to (5.32). Consider the rhombus R of $\mathbb{T}$ comprising all vertices of the form $m\mathbf{i}+n\mathbf{j}$ for $0 \leq m, n \leq 2a$. Let B be the event that R is traversed from left to right by a black path, and W the event that it is traversed from top to bottom by a white path. These two events are mutually exclusive with the same probability, and one or the other necessarily occurs. Therefore $P(B) = \frac{1}{2}$. On B, there exists a left–right crossing of the (sub-)rectangle $[a, 2a] \times [0, a\sqrt{3}]$, and the claim follows. □

Proof of Theorem 5.31. By (5.29) and Lemma 5.30, there exists $\alpha > 0$ such that

$$P(H_{8n,n\sqrt{3}}) \geq \alpha, \qquad n \geq 1.$$

We may represent the annulus A_n as the pairwise-intersection of six copies of $R_{8n,n\sqrt{3}}$ obtained by translation and rotation, as illustrated in Figure 5.11. If each of these is traversed by a black path in its long direction, than the event O_n occurs. By the FKG inequality,

$$P(O_n) \geq \alpha^6,$$

and the theorem is proved. □

5.6 The critical probability in two dimensions

We revert to bond percolation on the square lattice in this section. The square lattice has a property of self-duality, illustrated in Figure 3.1. 'Percolation of open edges on the primal lattice' is dual to 'percolation of closed edges on the dual lattice'. The self-dual value of p is thus $p = \frac{1}{2}$, and it was long been believed by physicists that the self-dual point is also the critical point p_c. Theodore Harris [121] proved by a geometric construction that $\theta(\frac{1}{2}) = 0$, whence $p_c(\mathbb{Z}^2) \geq \frac{1}{2}$. Harry Kesten [135] proved the complementary inequality.

(5.33) Theorem [121, 135]. *The critical probability of bond percolation on the square lattice equals* $\frac{1}{2}$. *Furthermore,* $\theta(\frac{1}{2}) = 0$.

Before giving a proof, we make some comments on the original proof. Harris [121] showed that, if $\theta(\frac{1}{2}) > 0$, then one can construct closed dual circuits around the origin. Such circuits prevent the cluster C from being infinite, and therefore $\theta(\frac{1}{2}) = 0$, a contradiction. Similar 'path-construction' arguments were developed by Russo [186] and Seymour–Welsh [195] in a proof that $p > p_c$ if and only if $\chi(1-p) < \infty$. This so-called 'RSW method' has acquired prominence in recent work on SLE (see Sections 5.5 and 5.7).

The complementary inequality $p_c(\mathbb{Z}^2) \leq \frac{1}{2}$ was proved by Kesten in [135]. More specifically, he showed that, for $p < \frac{1}{2}$, the probability of an open left–right crossing of the rectangle $[0, 2^k] \times [0, 2^{k+1}]$ tends to zero as $k \to \infty$. With the benefit of hindsight, one may view his argument as establishing a type of sharp-threshold theorem for the event in question.

The arguments that prove Theorem 5.33 may be adapted to certain other situations. For example, Wierman [211] has proved similarly that the critical probabilities of bond percolation on the hexagonal/triangular pair of lattices (see Figure 5.9) are the dual pair of values satisfying the star–triangle transformation. Russo [187] adapted the arguments to site percolation on the square lattice. It is easily seen by the same arguments[3] that site percolation on the triangular lattice has critical probability $\frac{1}{2}$.

The proof of Theorem 5.33 is broken into two parts.

Proof of Theorem 5.33: $\theta(\frac{1}{2}) = 0$, *and hence* $p_c \geq \frac{1}{2}$. Zhang discovered a beautiful proof of this, using only the uniqueness of the infinite cluster, see [95]. Set $p = \frac{1}{2}$. Let $T(n) = [0, n]^2$, and find N sufficiently large that

$$P_{\frac{1}{2}}(\partial T(n) \leftrightarrow \infty) > 1 - (\tfrac{1}{8})^4, \qquad n \geq N.$$

We set $n = N + 1$. Let A^l, A^r, A^t, A^b be the (respective) events that the left, right,

[3] See also Section 5.8.

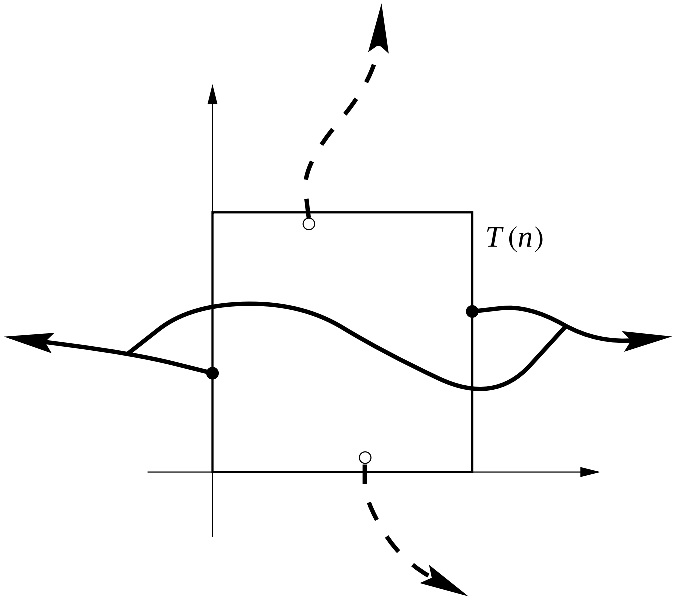

Figure 5.12. The left and right sides of the box $T(n)$ are joined to infinity by open paths of the primal lattice, and the top and bottom sides of the dual box $T(n)_{\mathrm{d}}$ are joined to infinity by closed dual paths. Using the uniqueness of the infinite open cluster, the two open paths must be joined by an open path. This forces the existence of two disjoint infinite closed clusters in the dual.

top, bottom sides of $T(n)$ are joined to ∞ off $T(n)$. By the FKG inequality,

$$\begin{aligned} P_{\frac{1}{2}}(T(n) \not\leftrightarrow \infty) &= P_{\frac{1}{2}}(\overline{A^{\mathrm{l}}} \cap \overline{A^{\mathrm{r}}} \cap \overline{A^{\mathrm{t}}} \cap \overline{A^{\mathrm{b}}}) \\ &\geq P_{\frac{1}{2}}(\overline{A^{\mathrm{l}}})P(\overline{A^{\mathrm{r}}})P(\overline{A^{\mathrm{t}}})P(\overline{A^{\mathrm{b}}}) \\ &= P_{\frac{1}{2}}(\overline{A^{\mathrm{g}}})^4 \end{aligned}$$

by symmetry, for $\mathrm{g} = \mathrm{l}, \mathrm{r}, \mathrm{t}, \mathrm{b}$. Therefore

$$P_{\frac{1}{2}}(A^{\mathrm{g}}) \geq 1 - \big[1 - P_{\frac{1}{2}}(T(n) \leftrightarrow \infty)\big]^{1/4} > \tfrac{7}{8}.$$

We consider next the dual box, with vertex set $T(n)_{\mathrm{d}} = [0, n-1]^2 + (\frac{1}{2}, \frac{1}{2})$. Let $A_{\mathrm{d}}^{\mathrm{l}}$, $A_{\mathrm{d}}^{\mathrm{r}}$, $A_{\mathrm{d}}^{\mathrm{t}}$, $A_{\mathrm{d}}^{\mathrm{b}}$ denote the (respective) events that the left, right, top, bottom sides of $T(n)_{\mathrm{d}}$ are joined to ∞ by a *closed* dual path off $T(n)_{\mathrm{d}}$. Since each edge of the dual is closed with probability $\frac{1}{2}$,

$$P_{\frac{1}{2}}(A_{\mathrm{d}}^{\mathrm{g}}) > \tfrac{7}{8}, \qquad \mathrm{g} = \mathrm{l}, \mathrm{r}, \mathrm{t}, \mathrm{b}.$$

Consider the event $A = A^{\mathrm{l}} \cap A^{\mathrm{r}} \cap A_{\mathrm{d}}^{\mathrm{t}} \cap A_{\mathrm{d}}^{\mathrm{b}}$, illustrated in Figure 5.12. Clearly, $P_{\frac{1}{2}}(\overline{A}) \leq \frac{1}{2}$, so that $P_{\frac{1}{2}}(A) \geq \frac{1}{2}$. However, on A, either $\mathbb{L}^2$ has two infinite

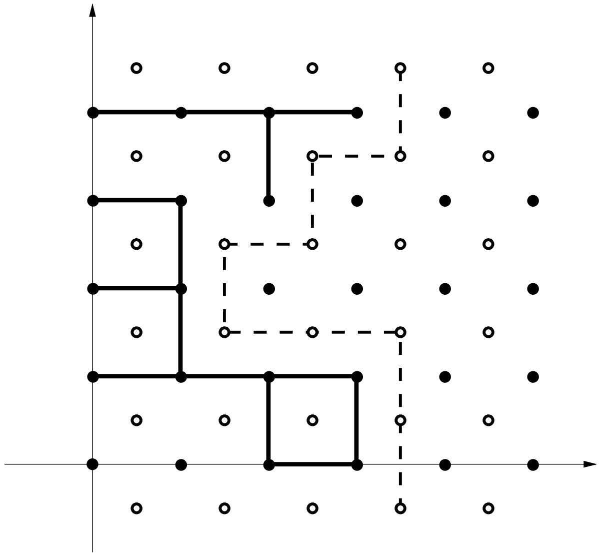

Figure 5.13. If there is no open left–right crossing of $S(n)$, there must exist a closed top–bottom crossing in the dual.

open clusters, or its dual has two infinite closed clusters. By Theorem 5.22, each event has probability 0, a contradiction. We deduce that $\theta(\frac{1}{2}) = 0$, implying in particular that $p_c \geq \frac{1}{2}$. □

Proof of Theorem 5.33: $p_c \leq \frac{1}{2}$. We give two proofs. The first uses the general exponential-decay Theorem 5.1. The second is previously unpublished work of Stanislav Smirnov, and is close in spirit to Kesten's original proof.

Proof A. Suppose instead that $p_c > \frac{1}{2}$. By Theorem 5.1, there exists $\gamma > 0$ such that

$$P_{\frac{1}{2}}(0 \leftrightarrow \partial[-n, n]^2) \leq e^{-\gamma n}, \qquad n \geq 1. \tag{5.34}$$

Let $S(n)$ be the graph with vertex set $[0, n+1] \times [0, n]$ and edge set containing all edges inherited from $\mathbb{L}^2$ except those in either the left side or the right side of $S(n)$. Let A be the event that there exists an open path joining the left side and right side of $S(n)$. If A does not occur, then the top side of the dual of $S(n)$ is joined to the bottom side by a closed dual path. Since the dual of $S(n)$ is isomorphic to $S(n)$, and since $p = \frac{1}{2}$, it follows that $P_{\frac{1}{2}}(A) = \frac{1}{2}$. See Figure 5.13. However, by (5.34),

$$P_{\frac{1}{2}}(A) \leq (n+1)e^{-\gamma n},$$

a contradiction for large n. We deduce that $p_c \leq \frac{1}{2}$.

Proof B. Let $\Lambda_r = [-r, r]^2$, and $A_r = \Lambda_{3r} \setminus \Lambda_r$ be an 'annulus'. The principal ingredient is an estimate that follows from the square-lattice version[4] of the RSW

[4]See Exercise 5.5.

Theorem 5.31. There exist $c' \log r$ disjoint annuli Λ_k within $[-r, r]^2$, and each of these contains a dual closed circuit with probability at least $\sigma > 0$. Therefore, $g(r) = P_{\frac{1}{2}}(0 \leftrightarrow \partial\Lambda_r)$ satisfies

(5.35)
$$g(r) \le (1-\sigma)^{c' \log r} = r^{-\alpha},$$

where $c', \alpha > 0$.

There is a variety of ways of implementing the basic argument of this proof, of which we choose the following. Let $R_n = [0, 2n] \times [0, n]$ where $n \ge 1$, and let H_n be the event that R_n is traversed by an open path from left to right. In the above notation, $P_{\frac{1}{2}}(A) = \frac{1}{2}$, and hence, by Lemma 5.28 rewritten for the square lattice, there exists $\gamma > 0$ such that

(5.36)
$$P_{\frac{1}{2}}(H_n) \ge \gamma, \qquad n \ge 1.$$

We take $p \ge \frac{1}{2}$ and work with the dual model. Let S_n be the dual box $(\frac{1}{2}, \frac{1}{2}) + [0, 2n-1] \times [0, n+1]$, and let V_n be the event that S_n is traversed from top to bottom by a closed dual path. Let N_n be the number of pivotal edges for the event V_n, and let Π be the event that $N_n \ge 1$ and all pivotal edges are closed (in the dual). We claim that

(5.37)
$$P_p(N_n = k-1 \mid \Pi) \le kg\big((n-k)/k\big), \qquad 1 \le k < \tfrac{1}{2}n,$$

of which the proof follows.

For any top–bottom path l of V_n, we write $L(l)$ (respectively, $R(l)$ for the set of edges of S_n lying to the 'left' (respectively, 'right') of l. On Π, there exists a closed top–bottom path of S_n, and from amongst such paths we may pick the leftmost, denoted λ. As in the proof of Lemma 5.28, λ is measurable on the states of edges in and to the left of λ; that is to say, for any admissible l, the event $\{\lambda = l\}$ depends only on the states of edges in $l \cup L(l)$. Suppose that $\lambda = l$, so that every pivotal edge for V_n necessarily lies in l. We now take a walk along l from bottom to top, encountering in order the edges of l. Let $f_1, f_2, \dots, f_k$ be the pivotal edges thus encountered, with $f_i = \langle x_i, y_i \rangle$, and let y_0 be the initial vertex of l and x_{k+1} the last. For $i = 0, 1, \dots, k$, there exists a closed path of $R(l)$ from y_i to x_{i+1}. See Figure 5.14.

Suppose $N_n = k-1$ for some $k < \frac{1}{2}n$. As we move along l, we progressively reveal the closed clusters C_i of the y_i. That is, we first observe C_0, a cluster that contains an open path of $R(l)$ joining y_0 to x_1. Then we observe C_1, a cluster containing a path of $R(l)$ from y_1 to x_2, and so on. There exists j such that the L^∞-distance between y_j and x_{j+1} is at least $(n-k)/k$. Therefore, in this sequence of observations, there exists j such that C_j contains a path of $R(l)$ from y_j to $y_j + \partial\Lambda_{(n-k)/k}$. Conditional on the history of the process up to step j, the chance of this is no greater than $g((n-k)/k)$. Inequality (5.37) follows.

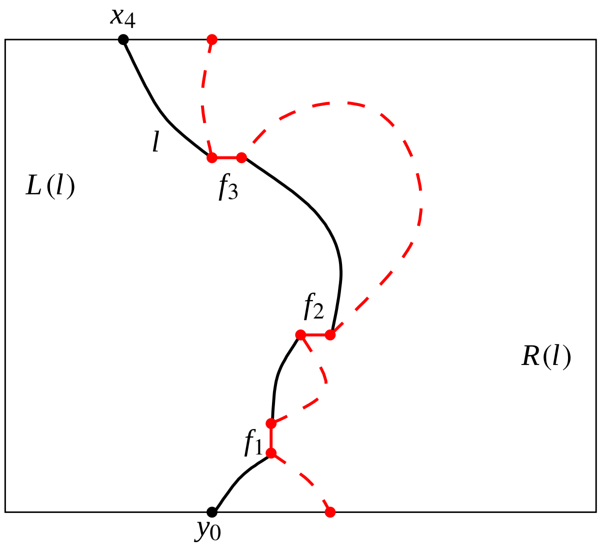

Figure 5.14. Between any two successive pivotal edges of the top–bottom crossing l, there exists a closed path joining their endpoints and (otherwise) lying entirely in $R(l)$. There are three pivotal edges f_i in this illustration, and the dashed lines are the closed connections of $R(l)$ joining successive f_i.

By (5.37) and (5.35),

$$\begin{aligned} E_p(N_n \mid \Pi) &\geq k \cdot P_p(N_n \geq k \mid \Pi) \\ &\geq k\left(1 - k^2\left(\frac{k}{n-k}\right)^{\alpha}\right) \end{aligned}$$

We choose $k = \frac{1}{2}n^{\beta}$ where $0 < \beta < \alpha/(2+\alpha)$, to obtain that

$$E_p(N_n \mid \Pi) \geq cn^{\beta}, \qquad n \geq 1, \tag{5.38}$$

where $c > 0$ is an absolute constant.

We prove next that

$$P_p(\Pi) \geq P_p(H_n)P_p(V_n). \tag{5.39}$$

Suppose V_n occurs, with $\lambda = l$, and let W_l be the event that there exists $e \in l$ with the following property: the dual edge $e^{\mathrm{d}} = \langle u, v\rangle$ has an endpoint, v say, that is joined to the right side of R_n by an open primal path of edges dual to edges in $R(l)$. By the definition of leftmost crossing, it is automatically the case that the other endpoint u is joined to the left side of R_n by a primal open path of edges dual to members of $L(l)$. Since $P_p(W_l \mid \lambda = l) \geq P_p(H_n)$,

$$P_p(H_n)P_p(V_n) \leq \sum_l P_p(\lambda = l)P_p(W_l \mid \lambda = l) = P_p(\Pi).$$

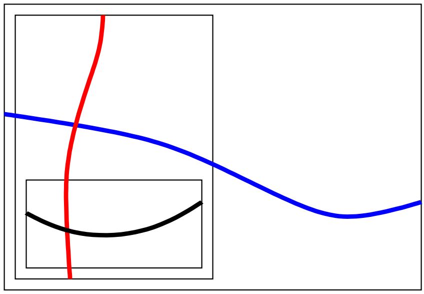

Figure 5.15. The boxes with aspect ratio 2 are arranged in such a way that, if all but finitely many are traversed in the long direction, then there must exist an infinite cluster

Since $P_p(H_n) = 1 - P_p(V_n)$, by (5.38) and Russo's formula (Theorem 4.75),

$$\frac{d}{dp} P_p(H_n) \geq E_p(N_n) \geq cn^\beta P_p(H_n)[1 - P_p(H_n)], \qquad p \geq \tfrac{1}{2}.$$

The resulting differential inequality

$$\left[\frac{1}{P_p(H_n)} - \frac{1}{1 - P_p(H_n)}\right] \frac{d}{dp} P_p(H_n) \geq cn^\beta$$

may be integrated over the interval $[\frac{1}{2}, p]$ to obtain[5] via (5.36) that

$$1 - P_p(H_n) \leq \frac{1}{\gamma} \exp\left\{-c(p - \tfrac{1}{2})n^\beta\right\}.$$

from which we extract the fact that

$$\sum_{n=1}^{\infty} [1 - P_p(H_n)] < \infty, \qquad p > \tfrac{1}{2}. \tag{5.40}$$

We now use a block argument[6] that was published in [58]. Consider the nested rectangles

$$B_{2r-1} = [0, 2^{2r}] \times [0, 2^{2r-1}], \quad B_{2r} = [0, 2^{2r}] \times [0, 2^{2r+1}], \qquad r \geq 1,$$

illustrated in Figure 5.15. Let K_{2r-1} (respectively, K_{2r}) be the event that B_{2r-1} (respectively, B_{2r}) is traversed from left to right (respectively, top to bottom) by an open path, so that $P_p(K_k) = P_p(H_{2^k})$. By (5.40) and the Borel–Cantelli lemma, all but finitely many of the K_k occur, P_p-almost surely. By Figure 5.15 again, this entails the existence of an infinite open cluster, whence $\theta(p) > 0$ and $p_c \leq \frac{1}{2}$. □

[5] The same point may be reached using the theory of influence, as in Exercise 5.4.
[6] An alternative block argument may be found in Section 5.8.

5.7 Cardy formula

There is a rich physical theory of phase transitions in theoretical physics, and critical percolation is at the heart of this theory. The case of two dimensions is very special, in that methods of conformality and complex analysis, linked to predictions of conformal field theory, have given rise to a beautiful and universal vision for the nature of such singularities. This vision is both analytical and geometrical. Its proof has been one of the principal targets of probability theory and theoretical physics over recent decades. The "road-map" to the proof is now widely accepted, and many key ingredients have become clear. There remain some significant problems.

The principal ingredient of the theory is the SLE process introduced in Section 2.5. In a classical theorem of Löwner [155], one sees that a growing curve γ in $\mathbb{R}^2$ may be encoded via conformal maps g_t in terms of a so-called 'driving function' $b : [0, \infty) \to \mathbb{R}$. Oded Schramm [189] predicted that a variety of scaling limits of stochastic process in $\mathbb{R}^2$ may be formulated thus, with b chosen as a Brownian motion with an appropriate variance parameter κ. He gave a partial proof that LERW on $\mathbb{Z}^2$, suitably re-scaled, has limit SLE_2, and he indicated that UST has limit SLE_8 and percolation SLE_6.

These observations did not come out of the blue. There was considerable earlier speculation around the idea of conformality, and we highlight the statement by John Cardy of his formula [54], and the discussions of Michael Aizenman and others concerning possible invariance under conformal maps (see, for example, [4, 5, 141]).

Much has been achieved since Schramm's paper [189]. Stanislav Smirnov [196, 197] has proved that critical site percolation on the triangular lattice satisfies Cardy's formula, and his route to 'complete conformality' and SLE_6 has been verified, see [51, 52] and [209]. Many of the critical exponents for the model have now been calculated rigorously, namely

$$\beta = \tfrac{5}{36}, \ \gamma = \tfrac{43}{18}, \ \nu = \tfrac{4}{3}, \ \rho = \tfrac{48}{5}, \tag{5.41}$$

together with the 'two-arm' exponent $\frac{5}{4}$, see [146, 200]. On the other hand, it has not yet been possible to extend such results to other principal percolation models such as bond or site percolation on the square lattice (some extensions have proved possible, see [61] for example).

On a related front, Smirnov [198, 199] has proved convergence of the re-scaled cluster boundaries of the critical Ising model (respectively, the associated random-cluster model) on $\mathbb{Z}^2$ to SLE_3 (respectively, $\text{SLE}_{16/3}$). This will be extended in [62] to the Ising model on any so-called isoradial graph, that is, a graph embeddable in $\mathbb{R}^2$ in such a way that the vertices of any face lie on the circumference of some circle of given radius r.

The theory of SLE will soon constitute a book in its own right[7], and similarly for the theory of the several scaling limits that have now been proved. These

[7] See [143].

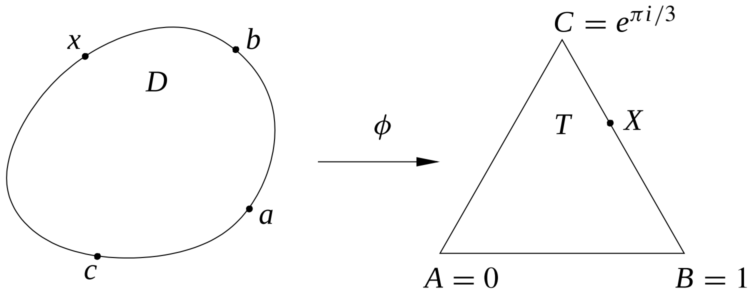

Figure 5.16. The conformal map ϕ is a bijection from D to the interior of T, and extends uniquely to the boundaries.

general topics are beyond the scope of the current work. We restrict ourselves here to the statement and proof of Cardy's formula for critical site percolation on the triangular lattice, and we make use of the accounts to be found in [209, 210]. See also [25, 44, 180].

We consider site percolation on the triangular lattice $\mathbb{T}$, with density $p = \frac{1}{2}$ of open (or 'black') vertices. It may be proved very much as in Theorem 5.33 that $p_c = \frac{1}{2}$ for this process, but this fact does not appear to be directly relevant to the material that follows. It is, rather, the 'self-duality' or 'self-matching' property that counts.

Let D ($\neq \mathbb{C}$) be an open simply connected domain in $\mathbb{R}^2$; for simplicity we shall assume that its boundary ∂D is a Jordan curve. Let a, b, c be distinct points of ∂D, taken in anticlockwise order around ∂D. There exists a conformal map ϕ from D to the interior of the equilateral triangle T of $\mathbb{C}$ with vertices $A = 0$, $B = 1$, $C = e^{\pi i/3}$, and such ϕ can be extended to the boundary ∂D in such a way that it becomes a homeomorphism from $D \cup \partial D$ (respectively, ∂D) to the closed triangle T (respectively, ∂T). There exists a unique such ϕ that maps a, b, c to A, B, C, respectively. With ϕ chosen accordingly, the image $X = \phi(x)$ of a fourth point $x \in \partial D$, taken for example on the arc from b to c, lies on the arc BC of T. See Figure 5.16.

The triangular lattice $\mathbb{T}$ is re-scaled to have mesh-size δ, and we ask about the probability $P_\delta(ac \leftrightarrow bx \text{ in } D)$ of an open path joining the arc ac to the arc bx, in an approximation to the intersection $(\delta\mathbb{T}) \cap D$ of the re-scaled lattice with D. It is a standard application of the RSW method of the last section to show that $P_\delta(ac \leftrightarrow bx \text{ in } D)$ is uniformly bounded away from 0 and 1 as $\delta \to 0$. It thus seems reasonable that this probability should converge as $\delta \to 0$, and Cardy's formula (together with conformality) tells us the value of the limit.

(5.42) Theorem. Cardy formula [54, 196, 197]. *In the notation introduced above,*

$$P_\delta(ac \leftrightarrow bx \text{ in } D) \to |BX| \qquad \text{as } \delta \to 0. \tag{5.43}$$

Some history: In [54], John Cardy stated the limit of $P_\delta(ac \leftrightarrow bx \text{ in } D)$ as a hypergeometric function of a certain cross-ratio. His derivation was based on

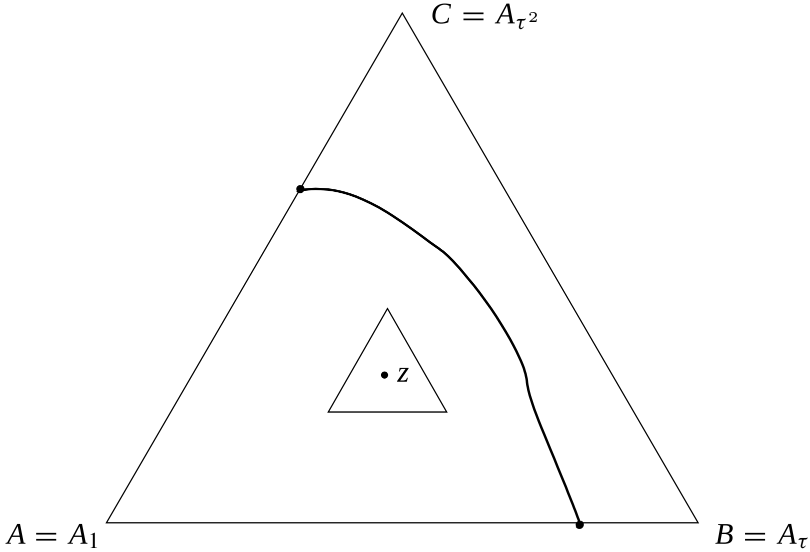

Figure 5.17. An illustration of the event $E_1^n(z)$, that z is separated from $A_\tau A_{\tau^2}$ by a black path joining $A_1 A_\tau$ and $A_1 A^{\tau^2}$.

arguments from conformal field theory, and was widely accepted as an inspired piece of physics. Lennart Carleson recognised the hypergeometric function in terms of the conformal map from a rectangle to a triangle, and was led to conjecture the simple form of (5.43). The limit was proved in 2001 by Stanislav Smirnov [196, 197]. The proof utilizes the three-way symmetry of the triangular lattice in a somewhat mysterious way.

The Cardy formula is, in a sense, only the beginning of the story of the scaling limit of critical two-dimensional percolation. It leads naturally to a full picture of the scaling limits of open paths, within the context of the Schramm–Löwner evolution SLE_6. One explicit application is towards the calculation of critical exponents [146, 200], but SLE_6 presents a much fuller picture than this. Further details may be found in [52, 53, 197, 209]. The principal open problem at the time of writing is to extend the scaling limit beyond site triangular model to either the bond or site model on another major lattice.

We prove Theorem 5.42 in the remainder of this section. This will be done first with $D = T$, the unit equilateral triangle, followed by the general case. Assume then that $D = T$ with T given as above. The vertices of T are $A = 0$, $B = 1$, $C = e^{i\pi/3}$. We take $\delta = 1/n$, and shall later let $n \to \infty$. Consider site percolation on $\mathbb{T}_n = (n^{-1}\mathbb{T}) \cap T$. One may draw either $\mathbb{T}_n$ or its dual graph $\mathbb{H}_n$, which comprises hexagons with centres at the vertices of $\mathbb{T}_n$. Each vertex of $\mathbb{T}_n$ (or equivalently, each face of $\mathbb{H}_n$) is declared *black* with probability $\frac{1}{2}$, and *white* otherwise. For ease of notation later, we write $A = A_1$, $B = A_\tau$, $C = A_{\tau^2}$, where

$$\tau = e^{2\pi i/3}.$$

For vertices V, W of T we write VW for the arc of the boundary of T from V to W.

Let z be the centre of a face of $\mathbb{T}_n$ (or equivalently, $z \in V(\mathbb{H}_n)$, the vertex-set of the dual graph $\mathbb{H}_n$). The events to be studied are as follows. Let $E_1^n(z)$ be the event that there exists a self-avoiding black path from A_1A_τ to $A_1A_{\tau^2}$ that separates z from $A_\tau A_{\tau^2}$. Let $E_\tau^n(z)$, $E_{\tau^2}^n(z)$ be given similarly after rotating the triangle clockwise by τ and τ^2, respectively. The event $E_1^n(z)$ is illustrated in Figure 5.17. We write

$$H_j^n(z) = P(E_j^n(z)), \qquad j = 1, \tau, \tau^2.$$

(5.44) Lemma. *The functions H_j^n, $j = 1, \tau, \tau^2$, are uniformly Hölder on $V(\mathbb{H}_n)$, in that there exist absolute constants $c \in (0, \infty)$, $\epsilon \in (0, 1)$ such that*

$$|H_j^n(z) - H_j^n(z')| \le c|z - z'|^\epsilon, \qquad z, z' \in V(\mathbb{H}_n), \tag{5.45}$$

$$1 - H_j^n(z) \le c|z - A_j|^\epsilon, \qquad z \in V(\mathbb{H}_n). \tag{5.46}$$

where A_j is interpreted as the complex number at the vertex A_j.

The domain of the H_j^n may be extended as follows: the set $V(\mathbb{H}_n)$ may be viewed as the vertex-set of a triangulation of a region slightly smaller than T, on each triangle of which H_j^n may be defined by linear interpolation between its values at the three vertices. Finally, the H_j^n may be extended up to the boundary of T in such a way that the resulting functions satisfy (5.45) for all $z, z' \in T$, and

$$H_j(A_j) = 1, \qquad j = 1, \tau, \tau^2. \tag{5.47}$$

Proof. It suffices to prove (5.45) for small $|z - z'|$. Suppose that $|z - z'| \le \frac{1}{100}$, say, and let F be the event that there exist both a black and a white circuit of the *entire* re-scaled triangular lattice $\mathbb{T}/n$, each of diameter smaller that $\frac{1}{4}$, and each having both z and z' in the bounded component of its complement. If F occurs, then either both or neither of the events $E_j^n(z)$, $E_j^n(z')$ occur, whence

$$|H_j^n(z) - H_j^n(z')| \le 1 - P(F).$$

When z and z' are a 'reasonable' distance from A_j, the white circuit prevents the occurrence of one of these events without the other. The black circuit is needed when z, z' are close to A_j.

There exists $C > 0$ such that one may find $\log(C/|z - z'|)$ vertex-disjoint annuli, each containing z, z' in their central 'hole', and each within distance $\frac{1}{8}$ of both z and z' (the definition of annulus precedes Theorem 5.31). By Theorem 5.31, the chance that no such annulus contains a black (respectively, white) circuit is no greater than

$$(1 - \sigma)^{\log(C/|z-z'|)}$$

whence $1 - P(F) \le c|z - z'|^\epsilon$ for suitable c and ϵ. Inequality (5.46) follows similarly with $z' = A_j$. ☐

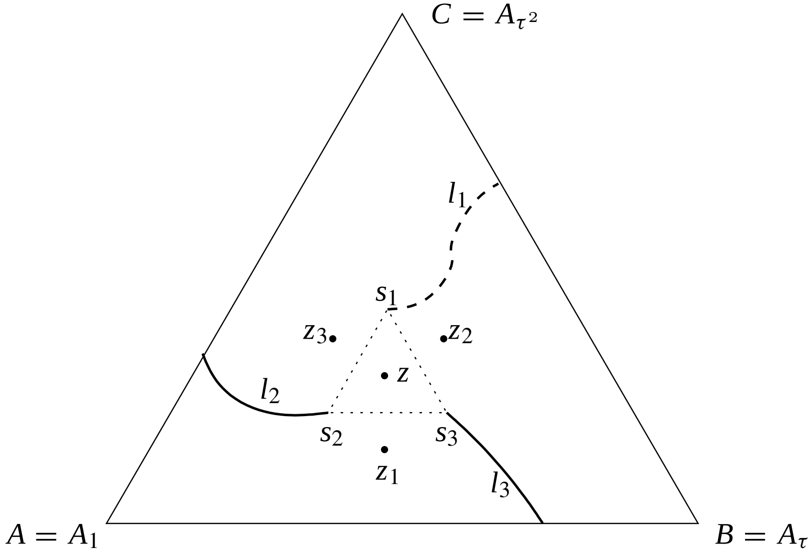

Figure 5.18. An illustration of the event $E_1^n(z_1) \setminus E_1^n(z)$. The path l_1 is white, and l_2, l_3 are black.

It is convenient to work in the space of uniformly Hölder functions on the closed triangle T satisfying (5.45)–(5.46). By the Arzelà–Ascoli theorem (see, for example, [67, Sect. 2.4]), this space is relatively compact. Therefore, the sequence of triples $(H_1^n, H_\tau^n, H_{\tau^2}^n)$ possesses subsequential limits in the sense of uniform convergence, and we shall see that any such limit is of the form $(H_1, H_\tau, H_{\tau^2})$ where the H_j are harmonic with certain boundary conditions, and satisfy (5.45)–(5.46). The boundary conditions guarantee the uniqueness of the H_j, and it will follow that $H_j^n \to H_j$ as $n \to \infty$.

We shall see in particular that

$$H_{\tau^2}(z) = \frac{2}{\sqrt{3}} |\Im(z)|,$$

the normalized imaginary part of z. The values of H_1 and H_τ are found by rotation. The claim of the theorem will follow by letting $z \to X \in BC$.

Let $(H_1, H_\tau, H_{\tau^2})$ be a subsequential limit as above. That the H_j are harmonic will follow from the fact that the functions

$$G_1 = H_1 + H_\tau + H_{\tau^2}, \qquad G_2 = H_1 + \tau H_\tau + \tau^2 H_{\tau^2}, \tag{5.48}$$

are analytic, and this analyticity will be implied by Morera's theorem on checking that the contour integrals of G_1, G_2 around triangles of a certain form are zero. The integration step amounts to summing the $H_j(z)$ over certain z and using a cancellation property that follows from the next lemma.

Let z be the centre of a face of $\mathbb{T}_n$, and let z_1, z_2, z_3 be the centres of the neighbouring faces ordered anticlockwise around z. See Figure 5.18.

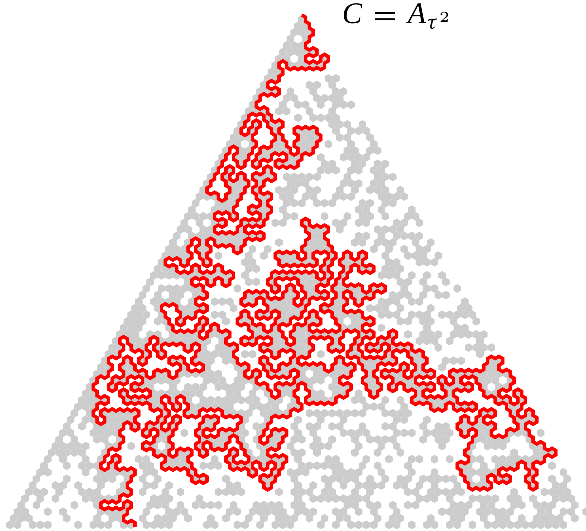

Figure 5.19. The exploration path η_n started at the top vertex A_{τ^2} and stopped when it hits the bottom side $A_1 A_\tau$ of the triangle.

(5.49) Lemma. *We have that*

$$P[E_1^n(z_1) \setminus E_1^n(z)] = P[E_\tau^n(z_2) \setminus E_\tau^n(z)] = P[E_{\tau^2}^n(z_3) \setminus E_{\tau^2}^n(z)].$$

Before proving this, we introduce the *exploration process* illustrated in Figure 5.19. Suppose that all vertices 'just outside' the arc $A_1 A_{\tau^2}$ (respectively, $A_\tau A_{\tau^2}$) of $\mathbb{T}_n$ are black (respectively, white). The exploration path is defined to be the unique path η_n on the edges of the dual (hexagonal) graph, beginning immediately above A_{τ^2} and descending to $A_1 A_\tau$ such that: as one traverses η_n from top to bottom, the vertex immediately on one's left (respectively, right), looking along the path from A_{τ^2}, is white (respectively, black). When traversing η_n thus, there is a white path on one's left and a black path on one's right.

Proof. The event $E_1^n(z_1) \setminus E_1^n(z)$ occurs if and only if there exist disjoint paths l_1, l_2, l_3 of $\mathbb{T}_n$ such that:

(i) l_1 is white and joins s_1 to $A_\tau A_{\tau^2}$,
(ii) l_2 is black and joins s_2 to $A_1 A_{\tau^2}$,
(iii) l_3 is black and joins s_3 to $A_1 A_\tau$.

See Figure 5.18 for an explanation of the notation. On this event, the exploration path η_n of Figure 5.19 passes through z and arrives at z along the edge $\langle z_3, z\rangle$ of $\mathbb{H}_n$. Furthermore, up to the time at which it hits z, it lies in the region of $\mathbb{H}_n$ between l_2 and l_1. Indeed we may take l_2 (respectively, l_1) to be the maximal black path (respectively, white path) of $\mathbb{T}_n$ lying on the right side (respectively, left side) of η_n up to this point.

Conditional on the event above, and with l_1 and l_2 given in terms of η_n accordingly, the states of vertices of $\mathbb{T}_n$ lying below $l_1 \cup l_2$ are independent Bernoulli variables. Thus the conditional probability of a *black* path l_3 satisfying (iii) is the same as that of a *white* path. We make this measure-preserving change, and then we interchange the colours white/black to conclude that: $E_1^n(z_1) \setminus E_1^n(z)$ has the same probability as the event that there exist disjoint paths l_1, l_2, l_3 of $\mathbb{T}_n$ such that:

(i) l_1 is black and joins s_1 to $A_\tau A_{\tau^2}$,
(ii) l_2 is white and joins s_2 to $A_1 A_{\tau^2}$,
(iii) l_3 is black and joins s_3 to $A_1 A_\tau$.

This is precisely the event $E_\tau^n(z_2) \setminus E_\tau^n(z)$, and the lemma is proved. □

We use Morera's theorem in order to show the necessary analyticity. This theorem states that: if $f : R \to \mathbb{C}$ is continuous on the open region R, and $\oint_\gamma f\, dz = 0$ for all closed curves γ in R, then f is analytic. It is standard (see [185, p. 208]) that it suffices to consider triangles γ in R. One may in fact restrict oneself to equilateral triangles with one side parallel to the x-axis. This may be seen either by an approximation argument, or by an argument based on the threefold Cauchy–Riemann equations

$$\frac{\partial f}{\partial 1} = \frac{1}{\tau}\frac{\partial f}{\partial \tau} = \frac{1}{\tau^2}\frac{\partial f}{\partial \tau^2}, \tag{5.50}$$

where $\partial/\partial j$ means the derivative in the direction of the complex number j.

(5.51) Lemma. *Let Γ be an equilateral triangle contained in the interior of T with sides parallel to those of T. Then*

$$\begin{aligned}\oint_\Gamma H_1^n(z)\, dz &= \oint_\Gamma [H_\tau^n(z)/\tau]\, dz + \mathrm{O}(n^{-\epsilon})\\ &= \oint_\Gamma [H_{\tau^2}^n(z)/\tau^2]\, dz + \mathrm{O}(n^{-\epsilon}).\end{aligned}$$

Proof. Every triangular facet of $\mathbb{T}_n$ (that is, a triangular union of faces) points either upwards (in that its horizontal side is at its bottom) or downwards. Let Γ be an equilateral triangle contained in the interior of T with sides parallel to those of T, and assume that Γ points upwards (the same argument works for downward-pointing triangles). Let Γ^n be the subgraph of $\mathbb{T}_n$ lying within Γ, so that Γ_n is a triangular facet of $\mathbb{T}_n$. Let $\mathcal{D}^n$ be the set of downward-pointing faces of Γ^n. Let η be a vector of $\mathbb{R}^2$ such that: if z is the centre of a face of $\mathcal{D}_n$ then $z+\eta$ is the centre of a neighbouring face, that is $\eta \in \{i, i\tau, i\tau^2\}/(n\sqrt{3})$. Write

$$h_j^n(z,\eta) = P[E_j^n(z+\eta) \setminus E_j^n(z)].$$

By Lemma 5.49,

$$\begin{aligned}H_1^n(z+\eta) - H_1^n(z) &= h_1^n(z,\eta) - h_1^n(z+\eta,-\eta)\\ &= h_\tau^n(z,\eta\tau) - h_\tau^n(z+\eta,-\eta\tau).\end{aligned}$$

Now,

$$H_\tau^n(z+\eta\tau) - H_\tau^n(z) = h_\tau^n(z,\eta\tau) - h_\tau^n(z+\eta\tau,-\eta\tau),$$

and so there is a cancellation in

$$I_\eta^n = \sum_{z\in\mathcal{D}^n} [H_1^n(z+\eta) - H_1^n(z)] - \sum_{z\in\mathcal{D}^n} [H_\tau^n(z+\eta\tau) - H_\tau^n(z)] \tag{5.52}$$

of all terms except those of the form $h_\tau^n(z',-\eta\tau)$ for certain z' lying in faces of $\mathbb{T}_n$ that abut $\partial\Gamma^n$. There are $O(n)$ such z', and therefore, by Lemma 5.44,

$$|I_\eta^n| \le O(n^{1-\epsilon}). \tag{5.53}$$

Consider the sum

$$J^n = \frac{1}{n}(I_i^n + \tau I_{i\tau}^n + \tau^2 I_{i\tau^2}^n),$$

where I_j^n is an abbreviation for $I_{j/n\sqrt{3}}^n$ in (5.52). The terms of the form $H_j^n(z)$ in (5.52) contribute 0 to J^n, since each is multiplied by $(1+\tau+\tau^2)n^{-1} = 0$. The remaining terms of the form $H_j^n(z+\eta)$, $H_j^n(z+\eta\tau)$ mostly disappear also, and one is left only with terms $H_j^n(z')$ for certain z' at the centre of upwards-pointing faces of $\mathbb{T}^n$ abutting $\partial\Gamma^n$. For example, the contribution from z' if its face is at the bottom (but not the corner) of Γ^n is

$$\frac{1}{n}\big[(\tau+\tau^2)H_1^n(z') - (1+\tau)H_\tau^n(z')\big] = -\frac{1}{n}[H_1^n(z') - H_\tau^n(z')/\tau].$$

When z' is at the right (respectively, left) edge of Γ^n, one obtains the same term multiplied by τ (respectively, τ^2). In summary,

$$\oint_{\Gamma^n} [H_1^n(z) - H_\tau^n(z)/\tau]\,dz = -J^n + O(n^{-\epsilon}) = O(n^{-\epsilon}), \tag{5.54}$$

by (5.53), where the first $O(n^{-\epsilon})$ term covers the fact that the z in (5.54) is a continuous rather than discrete variable. Since Γ and Γ^n differ only around their boundaries, and the H_j^n are uniformly Hölder,

$$\oint_{\Gamma} [H_1^n(z) - H_\tau^n(z)/\tau]\,dz = O(n^{-\epsilon}) \tag{5.55}$$

and, by a similar argument,

$$\oint_{\Gamma} [H_1^n(z) - H_{\tau^2}^n(z)/\tau^2]\,dz = O(n^{-\epsilon}). \tag{5.56}$$

The lemma is proved. □

As remarked after the proof of Lemma (5.45), the sequence $(H_1^n, H_\tau^n, H_{\tau^2}^n)$ possesses subsequential limits, and it suffices for convergence to show that all such limits are equal. Let $(H_1, H_\tau, H_{\tau^2})$ be such a subsequential limit. By Lemma 5.51, the contour integrals of H_1, H_τ/τ, H_{τ^2}/τ^2 around any Γ are equal. Therefore, the contour integrals of the G_i in (5.48) around any Γ equal zero. By Morera's theorem [2, 185], G_1 and G_2 are analytic on the interior of T, and furthermore they may be extended by continuity to the boundary of T. In particular, G_1 is analytic and real-valued, whence G_1 is a constant. By (5.46), $G_1(z) \to 1$ as $z \to 0$, whence

$$H_1 + H_\tau + H_{\tau^2} \equiv 1 \quad \text{on } T.$$

Therefore, the real part of G_2 satisfies

$$\text{(5.57)} \qquad \text{Re}(G_2) = H_1 - \tfrac{1}{2}(H_\tau + H_{\tau^2}) = \tfrac{1}{2}(3H_1 - 1),$$

and similarly

$$\text{(5.58)} \qquad 2\text{Re}(G_2/\tau) = 3H_\tau - 1, \quad 2\text{Re}(G_2/\tau^2) = 3H_{\tau^2} - 1.$$

Since the H_j are the real parts of analytic functions, they are harmonic. It remains to verify the relevant boundary conditions, and we will concentrate on the function H_{τ^2}. There are two ways of doing this, of which the first specifies certain derivatives of the H_j along the boundary of T.

By continuity, $H_{\tau^2}(C) = 1$ and $H_{\tau^2} \equiv 0$ on AB. We claim that the horizontal derivative, $\partial H_{\tau^2}/\partial 1$, is 0 on $AC \cup BC$. Once this is proved, it follows that $H_{\tau^2}(z)$ is the unique harmonic function on T satisfying these boundary conditions, namely the function $2|\Im(z)|/\sqrt{3}$. The remaining claim is proved as follows. Since G_2 is analytic, it satisfies the threefold Cauchy–Riemann equations (5.50). By (5.57)–(5.58),

$$\text{(5.59)} \qquad \frac{\partial H_{\tau^2}}{\partial 1} = \frac{2}{3}\text{Re}\left(\frac{1}{\tau^2}\frac{\partial G_2}{\partial 1}\right) = \frac{2}{3}\text{Re}\left(\frac{1}{\tau^3}\frac{\partial G_2}{\partial \tau}\right) = \frac{\partial H_1}{\partial \tau}.$$

Now, $H_1 \equiv 0$ on BC, and BC has gradient τ, whence the right side of (5.59)[8] equals 0 on BC. The same argument holds on AC with H_1 replaced by H_τ.

The alternative is slightly simpler, see [25]. For $z \in T$, $G_2(z)$ is a convex combination of $1, \tau, \tau^2$, and thus maps T to the complex triangle T' with these three vertices. Furthermore, G_2 maps ∂T to $\partial T'$, and $G_2(A_j) = j$ for $j = 1, \tau, \tau^2$. Since G_2 is analytic on the interior of T, it is conformal, and there is a unique such conformal map with this boundary behaviour, namely that composed of a suitable dilation, rotation, and translation of T. This identifies G_2 uniquely, and the functions H_j also by (5.57)–(5.58).

This concludes the proof of the Cardy formula when the domain D is an equilateral triangle. The proof for general D is essentially the same, on noting that

[8] We need also that G_2 may be continued analytically beyond the boundary of T, see [210].

a conformal image of a harmonic function is harmonic. First, one approximates to the boundary of D by a cycle of the triangular lattice with mesh δ. That G_1 ($\equiv 1$) and G_2 are analytic is proved as before, and hence the corresponding limit functions H_1, H_τ, H_{τ^2} are each harmonic with appropriate boundary conditions. We now apply conformal invariance. By the Riemann mapping theorem, there exists a conformal map ϕ from the inside of D to the inside of T that may be extended uniquely to their boundaries, and that maps a (respectively, b, c) to A (respectively, B, C). The triple $(H_1 \circ \phi^{-1}, H_\tau \circ \phi^{-1}, H_{\tau^2} \circ \phi^{-1})$ solves the corresponding problem on T. We have seen that there is a unique such triple on T, given as above, and equation (5.43) is proved.

5.8 The critical probability via the sharp-threshold theorem

We use the sharp-threshold Theorem 4.77 to prove the following.

(5.60) Theorem [211]. *The critical probability of site percolation on the triangular lattice satisfies* $p_c = \frac{1}{2}$.

This may be proved in the same manner as Theorem 5.33, but we choose here to use the sharp-threshold theorem. This theorem provides a convenient 'package' for obtaining the steepness of a box-crossing probability, viewed as a function of p. Other means, more elementary and discovered earlier, may be used instead. These include: Kesten's original proof [135] for bond percolation on the square lattice, Russo's 'approximate zero–one law' [188], and, most recently, the proof of Smirnov presented in Section 5.6. Sharp-thresholds were first used in [42] in the current context, and later in [43, 89, 90]. The present proof may appear somewhat shorter than that of [42].

Proof. Let $\theta(p)$ denote the percolation probability on the triangular lattice $\mathbb{T}$. We have that $\theta(\frac{1}{2}) = 0$, just as in the proof of the corresponding lower bound for the critical probability $p_c(\mathbb{L}^2)$ in Theorem 5.33, and we say no more about this. Therefore, $p_c \geq \frac{1}{2}$.

Two steps remain. First, we shall use the sharp-threshold theorem to deduce that, when $p > \frac{1}{2}$, long rectangles are traversed with high probability in the long direction. Then we shall use that fact, within a block argument, to show that $\theta(p) > 0$.

Each vertex is declared *black* (or open) with probability p, and *white* otherwise. In the notation introduced just prior to Lemma 5.28, let $H_n = H_{16n, n\sqrt{3}}$ be the event that the rectangle $R_n = R_{16n, n\sqrt{3}}$ is traversed by a black path in the long direction. By Lemmas 5.28–5.30, there exists $\tau > 0$ such that

$$\phi_{\frac{1}{2}}(H_n) \geq \tau, \qquad n \geq 1. \tag{5.61}$$

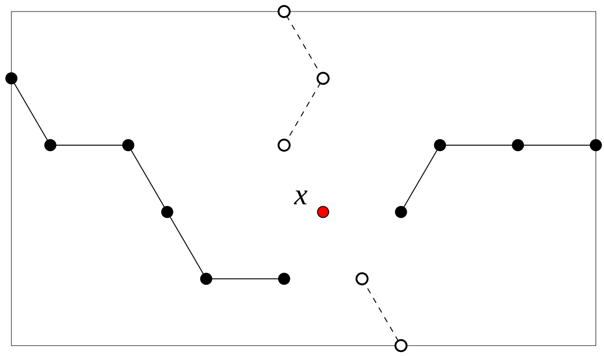

Figure 5.20. The vertex x is pivotal for H_n if and only if: there is left–right black crossing of R_n when x is black, and a top–bottom white crossing when x is white.

Let x be a vertex of R_n, and write $I_{n,p}(x)$ for the influence of x on the event H_n under the measure ϕ_p, see (4.24). Now, x is pivotal for H_n if and only if:

(i) the two vertical sides of R_n are connected by an open black path when x is black,

(ii) the two horizontal sides of R_n are connected by an open white path when x is white.

This event is illustrated in Figure 5.20.

Let $\frac{1}{2} \le p \le \frac{3}{4}$, say. By (ii),

$$(1-p)I_{n,p}(e) \le \phi_{1-p}(\mathrm{rad}(C_0) \ge n),$$

where

$$\mathrm{rad}(C_0) = \max\{|x| : 0 \leftrightarrow x\}$$

is the *radius* of the cluster at the origin. (Here, $|x|$ denotes the graph-theoretic distance from x to the origin.) Since $\theta(\frac{1}{2}) = 0$,

$$\phi_{1-p}(\mathrm{rad}(C_0) \ge n) \le \eta_n$$

where

$$\eta_n = \phi_{\frac{1}{2}}(\mathrm{rad}(C_0) \ge n) \to 0 \qquad \text{as } n \to \infty. \tag{5.62}$$

We have used the fact that $\theta(\frac{1}{2}) = 0$ here.

By (5.61) and Theorem 4.77, for large n,

$$\phi_p'(H_n) \ge c\tau(1 - \phi_p(H_n)) \log[1/(8\eta_n)], \qquad p \in [\tfrac{1}{2}, \tfrac{3}{4}],$$

which may be integrated to give

$$1 - \phi_p(H_n) \le (1-\tau)[8\eta_n]^{c\tau(p-\frac{1}{2})}, \qquad p \in [\tfrac{1}{2}, \tfrac{3}{4}]. \tag{5.63}$$

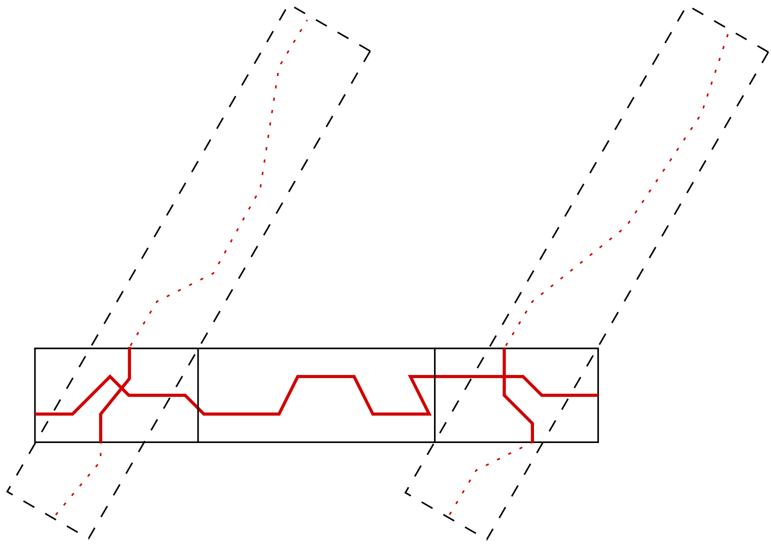

Figure 5.21. A block is declared 'red' if it contains open paths that: (i) traverse it in the long direction, and (ii) traverse it in the short direction within the $3n \times n\sqrt{3}$ region at each end of the block. The shorter crossings exist if the inclined blocks are traversed in the long direction.

Let $p > \frac{1}{2}$. By (5.62)–(5.63),

$$\phi_p(H_n) \to 1 \qquad \text{as } n \to \infty. \tag{5.64}$$

We turn to the required block argument, which differs from that of Section 5.6 in that we shall use no explicit estimate of $\phi_p(H_n)$. Roughly speaking, copies of the rectangle R_n are distributed about $\mathbb{T}$ in such a way that each copy corresponds to an edge of a re-scaled copy of $\mathbb{T}$. The detailed construction of this 'renormalized block lattice' is omitted from these notes, and we rely on Figure 5.22 for explanation. The 'blocks' (that is, the copies of R_n) are in one–one correspondence with the edges of $\mathbb{T}$, and thus we may label the blocks as B_e, $e \in \mathbb{E}_{\mathbb{T}}$. Each block intersects just ten other blocks.

Next we define a 'block event', that is, a certain event defined on the configuration within a block. The first requirement for this event is that the block be traversed by an open path in the long direction. We shall require some further paths in order that, when two such blocks intersect, their union contains a single component that traverses each in its long direction. In specific, we require open paths traversing the block in the short direction, within each of the two extremal $3n \times n\sqrt{3}$ regions of the block. A block is coloured *red* if the above paths exist within it. See Figure 5.21. If two red blocks, B_e and B_f say, are such that e and f share a vertex, then their union possesses a single open component containing paths traversing each of B_e and B_f.

If the block R_n fails to be red, then one or more of the blocks in Figure 5.21 is not traversed by an open path in the long direction. Therefore, $\rho_n := \phi_p(R_n \text{ is red})$

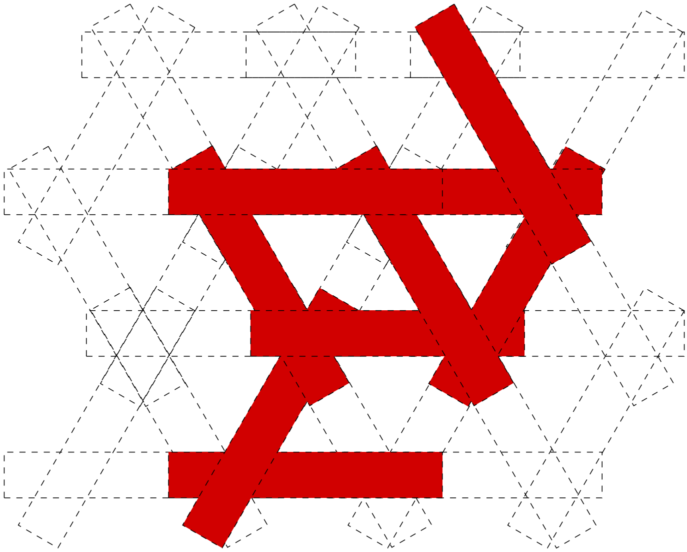

Figure 5.22. Each block is red with probability ρ_n. There is an infinite cluster of red blocks with strictly positive probability, and any such cluster contains an infinite open cluster of the original lattice.

satisfies

$$1 - \rho_n \le 3[1 - \phi_p(H_n)] \to 0 \qquad \text{as } n \to \infty, \tag{5.65}$$

by (5.64).

The states of different blocks are dependent random variables, but any collection of disjoint blocks have independent states. We shall count paths in the dual, as in (3.8), to obtain that there exists, with strictly positive probability, an infinite path in $\mathbb{T}$ comprising edges e such that every such B_e is red. This implies the existence of a infinite open cluster in the original lattice.

If the red cluster at the origin of the block lattice is finite, there exists a path in the dual lattice (a copy of the hexagonal lattice) that crosses only non-red blocks (as in Figure 3.2). Within any dual path of length m, there exists a set of $\lfloor m/12 \rfloor$ or more edges such that the corresponding blocks are pairwise disjoint. Therefore, the probability that the origin of the block lattice lies in a *finite* cluster only of red blocks is no greater than

$$\sum_{m=6}^{\infty} 3^m (1 - \rho_n)^{\lfloor m/12 \rfloor}.$$

By (5.65), this may be made smaller than $\frac{1}{2}$ by choosing n sufficiently large. Therefore, $\theta(p) > 0$ for $p > \frac{1}{2}$, and the theorem is proved. □

5.9 Exercises

5.1. [32] Consider bond percolation on $\mathbb{Z}^2$ with $p = \frac{1}{2}$, and define the radius of the open cluster C at the origin by $\mathrm{rad}(C) = \max\{n : 0 \leftrightarrow \partial[-n, n]^2\}$. Use the BK inequality to show that

$$\mathbb{P}_{\frac{1}{2}}\big(\mathrm{rad}(C) \ge n\big) \ge \frac{1}{2\sqrt{n}}.$$

5.2. Let D_n be the largest diameter (in the sense of graph theory) of the open clusters of bond percolation on $\mathbb{Z}^d$ that intersect the box $[-n, n]^d$. Show when $p < p_c$ that $D_n/\log n \to \alpha(p)$ almost surely and in L^p, for some $\alpha(p) \in (0, \infty)$.

5.3. Consider bond percolation on $\mathbb{L}^2$ with density p. Let T_n be the box $[0, n]^2$ with periodic boundary conditions, that is, we identify any pair (u, v), (x, y) satisfying: either $u = 0$, $x = n$, $v = y$, or $v = 0$, $y = n$, $u = x$. For given $m < n$, let A be the event that some translate of $[0, m]^2$ in T_n is crossed by an open path either from top to bottom, or from left to right. Using the theory of influence or otherwise, show that

$$\mathbb{P}_p(A) \ge 1 - \tfrac{1}{2}m^{-c(p-\frac{1}{2})}, \qquad p > \tfrac{1}{2}.$$

5.4. Consider site percolation on the triangular lattice $\mathbb{T}$, and let $B(n)$ be the ball of radius n centred at the origin. Use the RSW theorem to show that

$$P_{\frac{1}{2}}(0 \leftrightarrow \partial B(n)) \ge cn^{-\alpha}$$

for constants $c, \alpha > 0$.

Using the coupling of Section 3.3 or otherwise, deduce that $\theta(p) \le c'(p - \frac{1}{2})^\beta$ for $p > \frac{1}{2}$ and constants $c', \beta > 0$.

5.5. By adapting the arguments of Section 5.5 or otherwise, develop an RSW theory for bond percolation on $\mathbb{Z}^2$.

5.6. Let D be an open simply connected domain in $\mathbb{R}^2$ whose boundary ∂D is a Jordan curve. Let a, b, x, c be distinct points on ∂D taken in anticlockwise order. Let $P_\delta(ac \leftrightarrow bx)$ be the probability that, in site percolation on the re-scaled triangular lattice $\delta\mathbb{T}$ with density $\frac{1}{2}$, there exists an open path within $D \cup \partial D$ from some point on the arc ac to some point on bx. Show that $P_\delta(ac \leftrightarrow bx)$ is uniformly bounded away from 0 and 1 as $\delta \to 0$.

5.7. Let $f : D \to \mathbb{C}$ where D is an open simply-connected region of the complex plane. If f is C^1 and satisfies the threefold Cauchy–Riemann equations (5.50), show that f is analytic.

6

Contact Model

The contact process is a model for the spread of disease about the vertices of a graph. It has a property of duality that arises through the reversal of time. For a vertex-transitive graph such as the d-dimensional lattice, there is a multiplicity of invariant measures if and only if there is a strictly positive probability of an unbounded path of infection in space–time starting from a given vertex. This observation permits the use of methodology developed for the study of oriented percolation. When the underlying graph is a tree, the model has three distinct phases, termed extinction, weak survival, and strong survival. There is a continuous-time percolation model that differs from the contact model in that the time axis is undirected.

6.1 Stochastic epidemics

One of the simplest stochastic models for the spread of an epidemic is as follows. Consider a population of constant size $N + 1$ that is suffering from an infectious disease. We can model the spread of the disease as a Markov process. Let $X(t)$ be the number of healthy individuals at time t and suppose that $X(0) = N$. We assume that, if $X(t) = n$, then the probability of a new infection during $(t, t+h)$ is proportional to the number of possible encounters between ill folk and healthy folk. That is,

$$\mathbb{P}\big(X(t+h) = n-1 \,\big|\, X(t) = n\big) = \lambda n(N+1-n)h + \mathrm{o}(h) \qquad \text{as } h \downarrow 0.$$

In the simplest situation, we assume that nobody recovers. It is easy to show that

$$G(s,t) = \mathbb{E}(s^{X(t)}) = \sum_{n=0}^{N} s^n \mathbb{P}(X(t) = n)$$

satisfies

$$\frac{\partial G}{\partial t} = \lambda(1-s)\left(N\frac{\partial G}{\partial s} - s\frac{\partial^2 G}{\partial s^2}\right)$$

with $G(s,0) = s^N$. There is no simple way of solving this equation, though a lot of information is available about approximate solutions.

This epidemic model is over-simplistic through the assumptions that:

- the process is Markovian,
- there are only two states and no recovery,
- there is total mixing, in that the rate of spread is proportional to the product of the numbers of infectives and susceptibles.

In 'practice' (computer viruses apart), one infects only individuals in one's immediate (bounded) vicinity. The introduction of spatial relationships into such a model adds a major complication, and is achieved through so-called 'contact model' of Harris [122].

Let $G = (V, E)$ be a (finite or infinite) graph with bounded vertex-degrees. The contact model on G is a continuous-time Markov process on the state space $\Sigma = \{0,1\}^V$. A state is therefore a 0/1 vector $\xi = (\xi(x) : x \in V)$, where 0 represents the healthy state and 1 the ill state. There are two parameters: an infection rate λ and a recovery rate δ. Transition-rates are given informally as follows. Suppose that the state at time t is $\xi \in \Sigma$, and let $x \in V$. Then

$$\begin{aligned} \mathbb{P}(\xi_{t+h}(x) = 0 \mid \xi_t = \xi) &= \delta h + \mathrm{o}(h), && \text{if } \xi(x) = 1, \\ \mathbb{P}(\xi_{t+h}(x) = 1 \mid \xi_t = \xi) &= \lambda N_\xi(x) h + \mathrm{o}(h), && \text{if } \xi(x) = 0, \end{aligned}$$

where $N_\xi(x)$ is the number of neighbours of x that are infected in ξ:

$$N_\xi(x) = \bigl|\{y \in V : y \sim x,\ \xi(y) = 1\}\bigr|.$$

Thus, each ill vertex recovers at rate δ, and in the meantime infects any given neighbour at rate λ.

Care is needed when specifying a Markov process through its transition rates, especially when G is infinite, since then Σ is uncountable. We shall see in the next section that the contact model can be constructed via a countably infinite collection of Poisson processes. More general approaches to the construction of interacting particle processes are described in [148] and summarized in Section 10.1.

6.2 Coupling and duality

The contact model can be constructed in terms of families of Poisson processes. This representation is both informative and useful for what follows. For each $x \in V$ we draw a 'time-line' $[0, \infty)$. On the time-line $\{x\} \times [0, \infty)$ we place a Poisson point process D_x with intensity δ. For each *ordered* pair $x, y \in V$ of neighbours, we let $B_{x,y}$ be a Poisson point process with intensity λ. These processes are taken to be independent of each other, and we can assume without

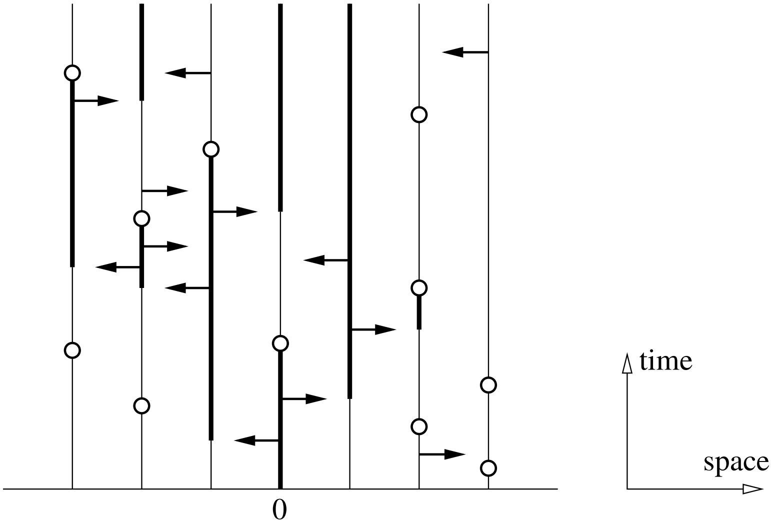

Figure 6.1. The so called 'graphical representation' of the contact process on the line $\mathbb{L}$. The horizontal line represents 'space', and the vertical line above a point x is the time-line at x. The marks $\circ$ are the points of cure, and the arrows are the arrows of infection. Suppose we are told that, at time 0, the origin is the unique infected point. In this picture, the initial infective is marked 0, and the bold lines indicate the portions of space–time which are infected.

loss of generality that the times occurring in the processes are distinct. Points in each D_x are called 'points of cure', and points in $B_{x,y}$ are called 'arrows of infection' from x to y. The appropriate probability measure is denoted by $\mathbb{P}_{\lambda,\delta}$.

The situation is illustrated in Figure 6.1 with $G = \mathbb{L}$. Let $(x, s), (y, t) \in V \times [0, \infty)$ where $s \le t$. We define a (*directed*) *path* from (x, s) to (y, t) to be a sequence $(x, s) = (x_0, t_0), (x_0, t_1), (x_1, t_1), (x_1, t_2), \dots, (x_n, t_{n+1}) = (y, t)$ with $t_0 \le t_1 \le \cdots \le t_{n+1}$, such that:

1. each interval $\{x_i\} \times [t_i, t_{i+1}]$ contains no points of D_{x_i},
2. $t_i \in B_{x_{i-1},x_i}$ for $i = 1, 2, \dots, n$.

We write $(x, s) \to (y, t)$ if there exists such a directed path.

We think of a point (x, u) of cure as meaning that an infection at x just prior to time u is cured at time u. A arrow of infection from x to y at time u means that an infection at x just prior to u is passed at time u to y. Thus, $(x, s) \to (y, t)$ means that y is infected at time t if x is infected at time s.

Let $\xi_0 \in \Sigma = \{0, 1\}^V$, and define $\xi_t \in \Sigma$, $t \in [0, \infty)$, by $\xi_t(y) = 1$ if and only if there exists $x \in V$ such that $\xi_0(x) = 1$ and $(x, 0) \to (y, t)$. It is clear that $(\xi_t : t \in [0, \infty))$ is a contact model with parameters λ and δ.

The above 'graphical representation' has several uses. First, it is a geometrical picture of the spread of infection that provides a coupling of contact models with all possible initial configurations ξ_0. Secondly, it provides couplings of contact models with different λ and δ, as follows. Let $\lambda_1 \le \lambda_2$ and $\delta_1 \ge \delta_2$, and consider the above representation with $(\lambda, \delta) = (\lambda_2, \delta_1)$. If we remove each point of cure with probability δ_2/δ_1 (respectively, each arrow of infection with probability

λ_1/λ_2), we obtain a representation of a contact model with parameters (λ_2, δ_2) (respectively, parameters (λ_1, δ_1)). We obtain thus that the passage of infection is non-increasing in δ and non-decreasing in λ.

There is a natural one–one correspondence between Σ and the power set 2^V of the vertex-set, given by $\xi \leftrightarrow I_\xi = \{x \in V : \xi(x) = 1\}$. We shall frequently regard vectors ξ as sets I_ξ. For $\xi \in \Sigma$ and $A \subseteq V$, we write ξ_t^A for the value of the contact model at time t starting at time 0 from the set A of infectives. It is immediate by the rules of the above coupling that:

(a) the coupling is *monotone* in that $\xi_t^A \subseteq \xi_t^B$ if $A \subseteq B$,
(b) the coupling is *additive* in that $\xi_t^{A\cup B} = \xi_t^A \cup \xi_t^B$.

(6.1) Theorem. Duality relation. *For* $A, B \subseteq V$,

$$\mathbb{P}_{\lambda,\delta}(\xi_t^A \cap B \neq \varnothing) = \mathbb{P}_{\lambda,\delta}(\xi_t^B \cap A \neq \varnothing). \tag{6.2}$$

Equation (6.2) can be written in the form

$$\mathbb{P}^A_{\lambda,\delta}(\xi_t \equiv 0 \text{ on } B) = \mathbb{P}^B_{\lambda,\delta}(\xi_t \equiv 0 \text{ on } A).$$

Proof. This hinges on the fact that a Poisson process remains a Poisson process when time is run backwards. The event on the left side of (6.2) is the union over $a \in A$ and $b \in B$ of the event that $(a, 0) \to (b, t)$. If we reverse the direction of time, and the directions of the arrows of infection, the probability of this event is unchanged and it corresponds now to the event on the right side of (6.2). □

6.3 Invariant measures and percolation

In this and the next section, we consider the contact model $\xi = (\xi_t : t \geq 0)$ when the underlying graph is the d-dimensional cubic lattice $\mathbb{L}^d$, with $d \geq 1$. Thus, ξ is a Markov process on the state space $\Sigma = \{0, 1\}^{\mathbb{Z}^d}$. Let $\mathcal{I}$ be the set of invariant measures of ξ, that is, the set of probability measures μ on Σ such that $\mu S_t = \mu$, where $S = (S_t : t \geq 0)$ is the transition semigroup of the process. It is elementary that $\mathcal{I}$ is a convex set of measures: if $\phi_1, \phi_2 \in \mathcal{I}$, then $\alpha\phi_1 + (1-\alpha)\phi_2 \in \mathcal{I}$ for $\alpha \in [0, 1]$. Therefore, $\mathcal{I}$ is determined by knowledge of the set $\mathcal{I}_\mathrm{e}$ of *extremal* invariant measures. A further discussion of the transition semigroup and its relationship to invariant measures can be found in Section 10.1.

The partial order on Σ induces a partial order on probability measures on Σ in the usual way, and we denote this by $\leq_{\mathrm{st}}$. It turns out that $\mathcal{I}$ possesses a 'minimal' and 'maximal' element, with respect to $\leq_{\mathrm{st}}$. The minimal measure (or 'lower invariant measure') is the measure that places probability 1 on the empty set, denoted $\delta_\varnothing$. It is called 'lower' because $\delta_\varnothing \leq_{\mathrm{st}} \mu$ for all measures μ on Σ.

The maximal measure (or 'upper invariant measure') is constructed as the weak limit of the contact model beginning with the set $\xi_0 = \mathbb{Z}^d$. Let μ_s denote the law of $\xi_s^{\mathbb{Z}^d}$. Since $\xi_s^{\mathbb{Z}^d} \subseteq \mathbb{Z}^d$,

$$\mu_0 S_s = \mu_s \leq_{\text{st}} \mu_0.$$

By the monotonicity of the coupling,

$$\mu_{s+t} = \mu_0 S_s S_t = \mu_s S_t \leq_{\text{st}} \mu_t,$$

whence the limit

$$\lim_{t\to\infty} \mu_t(f)$$

exists for any bounded increasing function $f : \Sigma \to \mathbb{R}$. Using the compactness of $(\Sigma, \mathcal{F})$ and a result from measure theory, the weak limit

$$\overline{\nu} = \lim_{t\to\infty} \mu_t$$

exists, and is called the *upper invariant measure*. It is clear by the method of its construction that $\overline{\nu}$ is invariant under the action of any translation of $\mathbb{L}^d$.

(6.3) Proposition. *We have that* $\delta_\varnothing \leq_{\text{st}} \nu \leq_{\text{st}} \overline{\nu}$ *for every* $\nu \in \mathcal{I}$.

Proof. Let $\nu \in \mathcal{I}$. The first inequality is trivial. Clearly, $\nu \leq_{\text{st}} \mu_0$, since μ_0 is concentrated on the maximal set $\mathbb{Z}^d$. By the monotonicity of the coupling,

$$\nu = \nu S_t \leq_{\text{st}} \mu_0 S_t = \mu_t, \qquad t \geq 0.$$

Let $t \to \infty$ to obtain that $\nu \leq_{\text{st}} \overline{\nu}$. □

By Proposition 6.3, there exists a unique invariant measure if and only if $\overline{\nu} = \delta_\varnothing$. In order to understand when this is so, we deviate briefly to consider a percolation-type question. Suppose we begin the process at a singleton, the origin say, and ask whether the probability of survival for all time is strictly positive. That is, we work with the percolation-type probability

$$\theta(\lambda, \delta) = \mathbb{P}_{\lambda,\delta}(\xi_t^0 \neq \varnothing \text{ for all } t \geq 0), \tag{6.4}$$

where $\xi_t^0 = \xi_t^{\{0\}}$. By a re-scaling of time, $\theta(\lambda, \delta) = \theta(\lambda/\delta, 1)$, and *we assume henceforth in his section that* $\delta = 1$, and we write $\mathbb{P}_\lambda = \mathbb{P}_{\lambda,1}$.

(6.5) Proposition. *The density of ill vertices under* $\overline{\nu}$ *equals* $\theta(\lambda)$. *That is,*

$$\theta(\lambda) = \overline{\nu}\big(\{\sigma \in \Sigma : \sigma_x = 1\}\big), \qquad x \in \mathbb{Z}^d.$$

Proof. The event that $\xi_T^0 \cap \mathbb{Z}^d \neq \varnothing$ is non-increasing in T, whence

$$\theta(\lambda) = \lim_{T\to\infty} \mathbb{P}_\lambda(\xi_T^0 \cap \mathbb{Z}^d \neq \varnothing).$$

By Proposition 6.1,

$$\mathbb{P}_\lambda(\xi_T^0 \cap \mathbb{Z}^d \neq \varnothing) = \mathbb{P}_\lambda(\xi_T^{\mathbb{Z}^d}(0) = 1),$$

and by weak convergence,

$$\mathbb{P}_\lambda(\xi_T^{\mathbb{Z}^d}(0) = 1) \to \overline{\nu}\big(\{\sigma \in \Sigma : \sigma_0 = 1\}\big).$$

The claim follows by the translation-invariance of $\overline{\nu}$. □

We define the critical value of the process by

$$\lambda_c = \lambda_c(d) = \sup\{\lambda : \theta(\lambda) = 0\}.$$

The function $\theta(\lambda)$ is non-decreasing, so that

$$\theta(\lambda) \begin{cases} = 0 & \text{if } \lambda < \lambda_c, \\ > 0 & \text{if } \lambda > \lambda_c. \end{cases}$$

By Proposition 6.5,

$$\overline{\nu} \begin{cases} = \delta_\varnothing & \text{if } \lambda < \lambda_c, \\ \neq \delta_\varnothing & \text{if } \lambda > \lambda_c. \end{cases}$$

The case $\lambda = \lambda_c$ is delicate, especially when $d \geq 2$, and it has been shown in [33], using a slab argument related to that of the proof of Theorem 5.17, that $\theta(\lambda_c) = 0$ for $d \geq 1$.

(6.6) Theorem [33]. *Consider the contact model on $\mathbb{L}^d$ with $d \geq 1$. The set $\mathcal{I}$ of invariant measures comprises a singleton if and only if $\lambda \leq \lambda_c$. That is, $\mathcal{I} = \{\delta_\varnothing\}$ if and only if $\lambda \leq \lambda_c$.*

There are further consequences of the arguments of [33] of which we mention one. The geometrical constructions of [33] enable a proof of the equivalent for the contact model of the 'slab' percolation Theorem 5.17. This in turn completes the proof, initiated in [69, 73], that the set of extremal invariant measures of the contact model on $\mathbb{L}^d$ is exactly $\mathcal{I}_e = \{\delta_\varnothing, \overline{\nu}\}$. See [71] also.

6.4 The critical value

This section is devoted to the following theorem[1]. Recall that the rate of cure is taken as $\delta = 1$.

[1] There are physical reasons to suppose that $\lambda_c(1) = 1.6494\ldots$, see the discussion of the so-called reggeon spin model in [91, 148].

(6.7) Theorem [122]. *For $d \ge 1$, we have that $(2d)^{-1} < \lambda_c(d) < \infty$.*

The lower bound is easily improved to $\lambda_c(d) \ge (2d-1)^{-1}$. The upper bound may be refined to $\lambda_c(d) \le d^{-1}\lambda_c(1) < 1$, as indicated in Exercise 6.2. See the accounts of the contact model in the two Liggett volumes [148, 150].

Proof. The lower bound is obtained by a random walk argument. The integer-valued process $N_t = |\xi_t^0|$ decreases by 1 at rate N_t. It increases by 1 at rate λT_t where T_t is the number of edges of $\mathbb{L}^d$ exactly one of whose endvertices x satisfies $\xi_t^0(x) = 1$. Now, $T_t \le 2dN_t$, and so the jump-chain of N_t is bounded above by a simple random walk $R = (R_n : n \ge 0)$ on $\{0, 1, 2, \dots\}$, with absorption at 0, and that moves to the right with probability

$$p = \frac{2d\lambda}{1 + 2d\lambda}$$

at each step. It is elementary that

$$\mathbb{P}(R_n = 0 \text{ for some } n \ge 0) = 1 \quad \text{if} \quad p \le \tfrac{1}{2},$$

and it follows that

$$\theta(\lambda) = 0 \quad \text{if} \quad \lambda < \frac{1}{2d}.$$

Just as in the case of percolation (Theorem 3.2) the upper bound on λ_c requires more work. Since $\mathbb{Z}^d$ may be embedded in $\mathbb{Z}$, it is elementary that $\lambda_c(d) \le \lambda_c(1)$. We show by a discretization argument that $\lambda_c(1) < \infty$. Let $\Delta > 0$, and let $m, n \in \mathbb{Z}$ be such that $m + n$ is even. We shall define independent random variables $X_{m,n}$ taking the values 0 and 1. We declare $X_{m,n} = 1$, and call (m, n) *open*, if and only if, in the graphical representation of the contact model, the following two events occur:

(a) there is no point of cure in the interval $\{m\} \times \big((n-1)\Delta, (n+1)\Delta\big]$,

(b) there exist left and right pointing arrows of infection from the interval $\{m\} \times \big(n\Delta, (n+1)\Delta\big]$.

It is immediate that the $X_{m,n}$ are independent, and

$$p = p(\Delta) = \mathbb{P}_\lambda(X_{m,n} = 1) = e^{-2\Delta}(1 - e^{-\lambda\Delta})^2.$$

We choose Δ to maximize $p(\Delta)$, which is to say that

$$e^{-\lambda\Delta} = \frac{1}{1+\lambda},$$

and

$$p = \frac{\lambda^2}{(1+\lambda)^{2+2/\lambda}}. \tag{6.8}$$

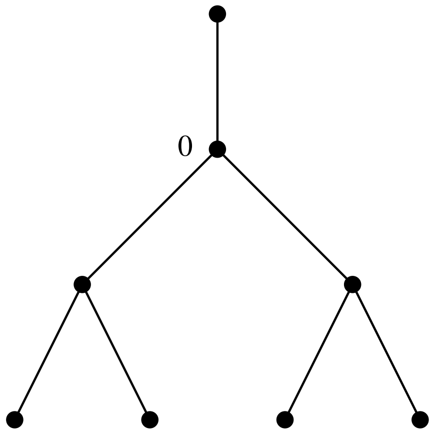

Figure 6.2. Part of the binary tree T_2.

Consider the $X_{m,n}$ as giving rise to a directed site percolation model on the first quadrant of a rotated copy of $\mathbb{Z}^2$. It can be seen that $\xi^0_{n\Delta} \supseteq B_n$, where B_n is the set of vertices of the form (m, n) that are reached from $(0, 0)$ along open paths of the percolation process. Now,

$$\mathbb{P}_\lambda\big(|B_n| = \infty \text{ for all } n \geq 0\big) > 0 \quad \text{if} \quad p > \vec{p}_c^{\,\text{site}}$$

where $\vec{p}_c^{\,\text{site}}$ is the critical probability of the percolation model. By (6.8),

$$\theta(\lambda) > 0 \quad \text{if} \quad \frac{\lambda^2}{(1+\lambda)^{2+2/\lambda}} > \vec{p}_c^{\,\text{site}}.$$

Since[2] $\vec{p}_c^{\,\text{site}} < 1$, the final inequality is valid for sufficiently large λ, and we have proved that $\lambda_c(1) < \infty$. □

6.5 The contact model on a tree

Let $d \geq 2$ and let T_d be the homogeneous (infinite) labelled tree in which every vertex has degree $d + 1$, illustrated in Figure 6.2. We identify a distinguished vertex, called the *origin* and denoted 0. Let $\xi = (\xi_t : t \geq 0)$ be a contact model on T_d with infection rate λ and initial state $\xi_0 = \{0\}$, and take $\delta = 1$.

With

$$\theta(\lambda) = \mathbb{P}_\lambda(\xi_t \neq \varnothing \text{ for all } t),$$

the process is said to *die out* if $\theta(\lambda) = 0$, and to *survive* if $\theta(\lambda) > 0$. It is said to *survive strongly* if

$$\mathbb{P}_\lambda\big(\xi_t(0) = 1 \text{ for unbounded times } t\big) > 0,$$

[2] *Exercise.*

and to *survive weakly* if it survives but it does not survive strongly. A process that survives weakly has the property that (with strictly positive probability) the illness exists for all time, but that (almost surely) there is a final time at which any given vertex is infected. It can be shown that weak survival never occurs on a lattice $\mathbb{L}^d$, see [150]. The picture is quite different on a tree.

The properties of survival and strong survival are evidently non-decreasing in λ, whence there exist values λ_c, λ_{ss} satisfying $\lambda_c \le \lambda_{ss}$ such that the process

$$\begin{array}{rll} \text{dies out} & \text{if} & \lambda < \lambda_c, \\ \text{survives weakly} & \text{if} & \lambda_c < \lambda < \lambda_{ss}, \\ \text{survives strongly} & \text{if} & \lambda > \lambda_{ss}. \end{array}$$

When is it the case that $\lambda_c < \lambda_{ss}$? The next theorems indicate that this occurs on T_d if $d \ge 6$. It was further proved in [171] that strict inequality holds whenever $d \ge 3$, and this was extended in [149] to $d \ge 2$. See [150, Chap. I.4] and the references therein.

(6.9) Theorem [171]. *For the contact model on the tree T_d with $d \ge 2$,*

$$\lambda_c < \frac{1}{d-1}.$$

(6.10) Theorem [171]. *For the contact model on the tree T_d with $d \ge 2$,*

$$\lambda_{ss} \ge \frac{1}{2\sqrt{d}}.$$

Proof of Theorem 6.9. Let $\rho \in (0,1)$, and $\nu_\rho(A) = \rho^{|A|}$ for any finite subset A of the vertex-set V of T_d. We shall observe the process $\nu_\rho(\xi_t)$. Let $g^A(t) = \mathbb{E}_\lambda^A(\nu_\rho(\xi_t))$. It is an easy calculation that

$$g^A(t) = |A|t\left[\frac{\nu_\rho(A)}{\rho}\right] + \lambda N t\big[\rho\nu_\rho(A)\big] + (1 - |A|t - \lambda N t)\nu_\rho(A) + \mathrm{o}(t), \tag{6.11}$$

as $t \downarrow 0$, where

$$N = \big|\{\langle x,y\rangle : x \in A,\ y \notin A\}\big|$$

is the number of edges of T_d with exactly one endvertex in A. Now,

$$N \ge (d+1)|A| - 2(|A|-1), \tag{6.12}$$

since there are no more than $|A| - 1$ edges having both endvertices in A. By (6.11),

$$\frac{d}{dt} g^A(t)\bigg|_{t=0} = (1-\rho)\left(\frac{|A|}{\rho} - \lambda N\right)\nu_\rho(A) \qquad (6.13)$$
$$\le (1-\rho)\nu_\rho(A)\left[\frac{|A|}{\rho}\big(1-\lambda\rho(d-1)\big) - 2\lambda\right]$$
$$\le -2\lambda(1-\rho)\nu_\rho(A) \le 0,$$

whenever

$$\lambda\rho(d-1) \ge 1. \qquad (6.14)$$

Assume that (6.14) holds. By (6.13) and the Markov property,

$$\frac{d}{du} g^A(u) = \mathbb{E}_\lambda^A\left(\frac{d}{dt} g^{\xi_u}(t)\bigg|_{t=0}\right) \le 0, \qquad (6.15)$$

implying that $g^A(u)$ is non-increasing in u.

With $A = \{0\}$, we have that $g(0) = \rho < 1$, and therefore $\lim_{t\to\infty} g(t) \le \rho$. On the other hand, if the process dies out, then (almost surely) $\xi_t = \varnothing$ for all large t, so that, by the bounded convergence theorm, $g(t) \to 1$ as $t \to \infty$. From this contradiction, we deduce that the process survives whenever there exists $\rho \in (0, 1)$ such that (6.14) holds. The theorem is proved. □

Proof of Theorem 6.10. Once again, take $\rho \in (0, 1)$. We draw the tree in the manner of Figure 6.3, and we let $l(x)$ be the generation number of the vertex x in this representation. For a finite subset A of V, let

$$w_\rho(A) = \sum_{x\in A} \rho^{l(x)},$$

with the convention that an empty summation equals 0.

As in (6.13), $h^A(t) = \mathbb{E}_\lambda^A(w_\rho(\xi_t))$ satisfies

$$\frac{d}{dt} h^A(t)\bigg|_{t=0} = \sum_{x\in A}\left(-\rho^{l(x)} + \lambda \sum_{\substack{y\in V:\ y\sim x,\\ y\notin A}} \rho^{l(y)}\right) \qquad (6.16)$$
$$\le -w_\rho(A) + \lambda \sum_{x\in A} \rho^{l(x)}[d\rho + \rho^{-1}]$$
$$= (\lambda d\rho + \lambda\rho^{-1} - 1) w_\rho(A).$$

Set

$$\rho = \frac{1}{\sqrt{d}}, \qquad \lambda = \frac{1}{2\sqrt{d}}, \qquad (6.17)$$

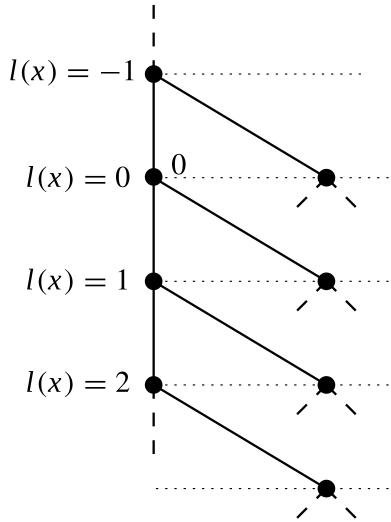

Figure 6.3. The binary tree T_2 'suspended' from a given doubly-infinite path, with the generation numbers as marked.

so that $\lambda d\rho + \lambda\rho^{-1} - 1 = 0$. By (6.16), $w_\rho(\xi_t)$ is a positive supermartingale. By the martingale convergence theorem, the limit

$$M = \lim_{t\to\infty} w_\rho(\xi_t), \tag{6.18}$$

exists $\mathbb{P}_\lambda^A$-almost surely. See Section 12.3 of [109] for an account of the convergence of martingales.

On the event $I = \{\xi_t(0) = 1 \text{ for unbounded times } t\}$, the process $w_\rho(\xi_t)$ changes its value (almost surely) by $\rho^0 = 1$ on an unbounded set of times t, in contradiction of (6.18). Therefore, $\mathbb{P}_\lambda^A(I) = 0$, and the process does not converge strongly under (6.17). The theorem is proved. □

6.6 Space–time percolation

The percolation models of Chapters 2 and 5 are discrete in that they inhabit a discrete graph $G = (V, E)$. There are a variety of *continuum* models of interest (see [99] for a summary) of which we distinguish the continuum model on $V \times \mathbb{R}$. One can consider this as the contact model with *undirected* time. We will encounter the related continuum random-cluster model in Chapter 9, together with its application to the quantum Ising model.

Let $G = (V, E)$ be a finite graph. The percolation model of this section inhabits the space $V \times \mathbb{R}$, which we refer to as space–time, and we think of $V \times \mathbb{R}$ as being obtained by attaching a 'time-line' $(-\infty, \infty)$ to each vertex $x \in V$.

Let $\lambda, \delta \in (0, \infty)$. The continuum percolation model on $V \times \mathbb{R}$ is constructed via processes of 'cuts' and 'bridges' as follows. For each $x \in V$, we select a Poisson process D_x of points in $\{x\} \times \mathbb{R}$ with intensity δ; the processes $\{D_x : x \in V\}$ are independent, and the points in the D_x are termed 'cuts'. For each $e = \langle x, y \rangle \in E$, we select a Poisson process B_e of points in $\{e\} \times \mathbb{R}$ with intensity λ; the processes $\{B_e : e \in E\}$ are independent of each other and of the D_x. Let $\phi_{\lambda,\delta}$ denote the probability measure associated with the family of such Poisson processes indexed by $V \cup E$.

For each $e = \langle x, y \rangle \in E$ and $(e, t) \in B_e$, we think of (e, t) as an edge joining the endpoints (x, t) and (y, t), and we refer to this edge as a 'bridge'. For $(x, s), (y, t) \in V \times \mathbb{R}$, we write $(x, s) \leftrightarrow (y, t)$ if there exists a path π with endpoints $(x, s), (y, t)$ such that: π is a union of cut-free sub-intervals of $V \times \mathbb{R}$ and bridges. For $\Lambda, \Delta \subseteq V \times \mathbb{R}$, we write $\Lambda \leftrightarrow \Delta$ if there exist $a \in \Lambda$ and $b \in \Delta$ such that $a \leftrightarrow b$.

For $(x, s) \in V \times \mathbb{R}$, let $C_{x,s}$ be the set of all points (y, t) such that $(x, s) \leftrightarrow (y, t)$. The clusters $C_{x,s}$ have been studied in [34], where the case $G = \mathbb{Z}^d$ was considered in some detail. Let 0 denote the origin $(0, 0) \in \mathbb{Z}^d \times \mathbb{R}$, and let $C = C_0$ denote the cluster at the origin. Noting that C is a union of line-segments, we write $|C|$ for its Lebesgue measure. The *radius* $\mathrm{rad}(C)$ of C is given by

$$\mathrm{rad}(C) = \sup\Big\{\|x\| + |t| : (x, t) \in C\Big\},$$

where

$$\|x\| = \sup_i |x_i|, \qquad x = (x_1, x_2, \dots, x_d) \in \mathbb{Z}^d,$$

is the supremum norm on $\mathbb{Z}^d$.

The critical point of the process is defined by

$$\lambda_c(\delta) = \sup\{\lambda : \theta(\lambda, \delta) = 0\},$$

where

$$\theta(\lambda, \delta) = \mathbb{P}_{\lambda,\delta}(|C| = \infty).$$

It is immediate by re-scaling time that $\theta(\lambda, \delta) = \theta(\lambda/\delta, 1)$, and we shall use the abbreviations $\lambda_c = \lambda_c(1)$ and $\theta(\lambda) = \theta(\lambda, 1)$.

(6.19) Theorem [34]. *Let $G = \mathbb{L}^d$ where $d \geq 1$, and consider continuum percolation on $\mathbb{Z}^d \times \mathbb{R}$.*

(a) *Let $\lambda, \delta \in (0, \infty)$. There exist γ, ν satisfying $\gamma, \nu > 0$ for $\lambda/\delta < \lambda_c$ such that:*

$$\mathbb{P}_{\lambda,\delta}(|C| \geq k) \leq e^{-\gamma k}, \qquad k > 0,$$
$$\mathbb{P}_{\lambda,\delta}(\mathrm{rad}(C) \geq k) \leq e^{-\nu k}, \qquad k > 0.$$

(b) *When* $d = 1$, $\lambda_c = 1$ *and* $\theta(1) = 0$.

There is a natural duality in $1 + 1$ dimensions (that is, when the underlying graph is the line $\mathbb{L}$), and it is easily seen in this case that the process is self-dual when $\lambda = \delta$. Part (b) identifies this self-dual point as the critical point. For general $d \geq 1$, the continuum percolation model on $\mathbb{L}^d \times \mathbb{R}$ has exponential decay of connectivity when $\lambda/\delta < \lambda_c$. The theorem is proved by an adaptation to the continuum of the methods used for $\mathbb{L}^{d+1}$. Theorem 6.19 will be useful for the study of the quantum Ising model in Section 9.4.

There has been considerable interest in the behaviour of the continuum percolation model on a graph G when the environment is itself chosen at random, that is, we take the $\lambda = \lambda_e$, $\delta = \delta_x$ to be random variables. More precisely, suppose that the Poisson process of cuts at a vertex $x \in V$ has some intensity δ_x, and that of bridges parallel to the edge $e = \langle x, y\rangle \in E$ has some intensity λ_e. Suppose further that the δ_x, $x \in V$, are independent, identically distributed random variables, and the λ_e, $e \in E$ also. Write Δ and Λ for independent random variables having the respective distributions, and P for the probability measure governing the environment. [As before, $\mathbb{P}_{\lambda,\delta}$ denotes the measure associated with the percolation model in the given environment. The above use of the letters Δ, Λ to denote random variables is temporary only.] The problem of understanding the behaviour of the system is now much harder, because of the fluctuations in intensities about G.

If there exist $\lambda', \delta' \in (0, \infty)$ such that $\lambda'/\delta' < \lambda_c$ and

$$P(\Lambda \leq \lambda') = P(\Delta \geq \delta') = 1,$$

then the process is almost surely dominated by the subcritical percolation process with parameters λ', δ', whence there is (almost sure) exponential decay in the sense of Theorem 6.19(i). This can fail in an interesting way if there is no such almost-sure domination, in that (under certain conditions) one can prove exponential decay in the space-direction but only a weaker decay in the time-direction. The problem arises since there will generally be regions of space that are favourable to the existence of large clusters, and other regions that are unfavourable. In a favourable region, there may be unnaturally long connections between two points with similar values for their time-coordinates.

For $(x, s), (y, t) \in \mathbb{Z}^d \times \mathbb{R}$ and $q \geq 1$, we define

$$d_q(x, s; y, t) = \max\big\{\|x - y\|, [\log(1 + |s - t|)]^q\big\}.$$

(6.20) Theorem [137, 138]. *Let* $G = \mathbb{L}^d$ *where* $d \geq 1$. *Suppose that*

$$K = \max\Big\{P\big([\log(1 + \Lambda)]^\beta\big), P\big([\log(1 + \Delta^{-1})]^\beta\big)\Big\} < \infty,$$

for some $\beta > 2d^2\big(1 + \sqrt{1 + d^{-1}} + (2d)^{-1}\big)$. *There exists* $Q = Q(d, \beta) > 1$ *such that the following holds. For* $q \in [1, Q)$ *and* $m > 0$, *there exists* $\epsilon =$

$\epsilon(d, \beta, K, m, q) > 0$ and $\eta = \eta(d, \beta, q) > 0$ such that: if

$$P\Big(\big[\log(1 + (\Lambda/\Delta))\big]^{\beta}\Big) < \epsilon,$$

there exist identically distributed random variables $D_x \in L^{\eta}(P)$, $x \in \mathbb{Z}^d$, such that

$$\mathbb{P}_{\lambda,\delta}\big((x, s) \leftrightarrow (y, t)\big) \le \exp\big[-m d_q(x, s; y, t)\big] \qquad \text{if } d_q(x, s; y, t) \ge D_x,$$

for $(x, s), (y, t) \in \mathbb{Z}^d \times \mathbb{R}$.

This version of the theorem of Klein can be found with explanation in [106]. It is proved by a so-called multiscale analysis.

Mention related results for contact models etc.

6.7 Exercises

6.1. Show that the critical probability of oriented site percolation on $\mathbb{L}^2$ satisfies $\vec{p}_c^{\text{site}} < 1$.

6.2. Let $d \ge 2$, and $\Pi : \mathbb{Z}^d \to \mathbb{Z}$ be given by

$$\Pi(x_1, x_2, \dots, x_d) = \sum_{i=1}^{d} x_i.$$

Let A_t denote a contact model on $\mathbb{Z}^d$ with parameter λ and starting at the origin. Show that one can couple A with a contact model C on $\mathbb{Z}$, with parameter λd and starting at the origin, in such a way that $\Pi(A_t) \supseteq C_t$ for all t.

Deduce that the critical point $\lambda_c(d)$ of the contact model on $\mathbb{L}^d$ satisfies $\lambda_c(d) \le d^{-1}\lambda_c(1)$.

6.3. [34] By adapting the corresponding argument for bond percolation on $\mathbb{L}^2$, or otherwise, show that the percolation probability of unoriented space-time percolation on $\mathbb{Z} \times \mathbb{R}$ satisfies $\theta(\lambda, \lambda) = 0$ for $\lambda > 0$.

7

Gibbs States

Brook's theorem states that a positive probability measure on a finite product may be decomposed into factors indexed by the cliques of its dependency graph. Closely related to this is the well known fact that a positive measure is a spatial Markov field if and only if it is a Gibbs state. The Ising and Potts models are introduced, and the n-vector model is mentioned.

7.1 Dependency graphs

Let $X = (X_1, X_2, \dots, X_n)$ be a family of random variables on a given probability space. For $i, j \in V = \{1, 2, \dots, n\}$ with $i \neq j$, we write $i \perp j$ if: X_i and X_j are independent *conditional* on $(X_k : k \neq i, j)$. The relation $\perp$ is thus symmetric, and it gives rise to a graph G with vertex set V and edge-set $E = \{\langle i, j\rangle : i \not\perp j\}$, called the *dependency graph* of X (or of its law). We shall see that the law of X may be expressed as a product over terms corresponding to complete subgraphs of G. A complete subgraph of G is called a *clique*, and we write $\mathcal{K}$ for the set of all cliques of G. For notational simplicity later, we designate the empty subset of V to be a clique, and thus $\varnothing \in \mathcal{K}$. A clique is *maximal* if no strict superset is a clique, and we write $\mathcal{M}$ for the set of maximal cliques of G.

We assume for simplicity that the X_i take values in some countable subset S of the reals $\mathbb{R}$. The law of X gives rise to a probability mass function π on S^n given by

$$\pi(\mathbf{x}) = P(X_i = x_i \text{ for } i \in V), \qquad \mathbf{x} = (x_1, x_2, \dots, x_n) \in S^n.$$

It is easily seen by the definition of independence that $i \perp j$ if and only if π may be factorized in the form

$$\pi(\mathbf{x}) = f(x_i, U)g(x_j, U), \qquad \mathbf{x} \in S^n,$$

for some functions f and g, where $U = (x_k : k \neq i, j)$. For $K \in \mathcal{K}$ and $\mathbf{x} \in S^n$, we write $\mathbf{x}_K = (x_i : i \in K)$. We call π *positive* if $\pi(\mathbf{x}) > 0$ for all $\mathbf{x} \in S^n$.

In the following, each function f_K acts on S^K.

(7.1) Theorem [49]. *Let π be a positive probability mass function on S^n. There exist functions $f_K : S^K \to [0,\infty)$, $K \in \mathcal{M}$, such that*

$$\pi(\mathbf{x}) = \prod_{K\in\mathcal{M}} f_K(\mathbf{x}_K), \qquad \mathbf{x}\in S^n. \tag{7.2}$$

In the simplest non-trivial example, let us assume that $i \perp j$ whenever $|i-j| \ge 2$. The maximal cliques are the pairs $\langle i, i+1\rangle$, and the mass function π may be expressed in the form

$$\pi(\mathbf{x}) = \prod_{i=1}^{n-1} f_i(x_i, x_{i+1}), \qquad \mathbf{x}\in S^n,$$

so that X is a Markov chain, whatever the direction of time.

Proof. We shall show that π may be expressed in the form

$$\pi(\mathbf{x}) = \prod_{K\in\mathcal{K}} f_K(\mathbf{x}_K), \qquad \mathbf{x}\in S^n, \tag{7.3}$$

for suitable f_K. Representation (7.2) follows from (7.3) by associating each f_K with some maximal clique K' that contains the clique K as a subset.

A representation of π in the form

$$\pi(\mathbf{x}) = \prod_r f_r(\mathbf{x})$$

is said to *separate* i and j if every f_r is a constant function of either x_i or x_j, that is, no f_r depends non-trivially on both x_i and x_j. Let

$$\pi(\mathbf{x}) = \prod_{A\in\mathcal{A}} f_A(\mathbf{x}_A) \tag{7.4}$$

be a factorization of π for some family $\mathcal{A}$ of subsets of V, and suppose that i, j satisfies: $i \perp j$, but i and j are not separated in (7.4). We shall construct from (7.4) a factorization that separates every pair r, s that is separated in (7.4), and in addition separates i, j. Continuing by iteration, we obtain a factorization that separates every pair i, j satisfying $i \perp j$, and this has the required form (7.3).

Since $i \perp j$, π may be expressed in the form

$$\pi(\mathbf{x}) = f(x_i, U)g(x_j, U) \tag{7.5}$$

for some f, g, where $U = (x_k : j \ne i, j)$. Fix $s \in S$, and write $h\big|_s$ for the function $h(\mathbf{x})$ evaluated with $x_j = s$. By (7.4),

$$\pi(\mathbf{x}) = \pi(\mathbf{x})\big|_s \frac{\pi(\mathbf{x})}{\pi(\mathbf{x})\big|_s} = \left(\prod_{A\in\mathcal{A}} f_A(\mathbf{x}_A)\big|_s\right)\frac{\pi(\mathbf{x})}{\pi(\mathbf{x})\big|_s}, \tag{7.6}$$

and, by (7.5),

$$\frac{\pi(\mathbf{x})}{\pi(\mathbf{x})\big|_s} = \frac{g(x_j, U)}{g(s, U)}$$

is independent of x_i. Equation (7.6) is thus the required representation, and the claim is proved. □

7.2 Markov fields and Gibbs states

Let $G = (V, E)$ be a finite graph, taken for simplicity without loops or multiple edges. Within statistics and statistical mechanics, there has been a great deal of interest in probability measures having a type of 'spatial Markov property' given in terms of the neighbour relation of G. We shall restrict ourselves here to measures on the sample space $\Sigma = \{0, 1\}^V$, while noting that the following results may extended without material difficulty to a larger product S^V where S is finite or countably infinite.

The vector $\sigma \in \Sigma$ may be placed in one–one correspondence with the subset $\eta(\sigma) = \{v \in V : \sigma_v = 1\}$ of V, and we shall use this correspondence freely. For any $W \subseteq V$, we define the *external boundary*

$$\Delta W = \{v \in V : v \notin W,\ v \sim w \text{ for some } w \in W\}.$$

For $s = (s_v : v \in V) \in \Sigma$, we write s_W for the sub-vector $(s_w : w \in W)$. We refer to the configuration of vertices in W as the 'state' of W.

(7.7) Definition. A probability measure π on Σ is said to be *positive* if $\pi(\sigma) > 0$ for all $\sigma \in \Sigma$. It is called a *Markov field* if it is positive and: for all $W \subseteq V$, conditional on the state of $V \setminus W$, the law of the state of W depends only on the state of ΔW. That is, π satisfies the *global Markov property*:

$$\text{(7.8)} \qquad \pi\big(\sigma_W = s_W \,\big|\, \sigma_{V\setminus W} = s_{V\setminus W}\big) = \pi\big(\sigma_W = s_W \,\big|\, \sigma_{\Delta W} = s_{\Delta W}\big),$$

for all $s \in \Sigma$, and $W \subseteq V$.

In the language of the previous section, π is a Markov field if and only if it is positive and its dependency graph is a subgraph of G. The key result about such measures is their representation in terms of a 'potential function' ϕ, in a form known as a 'Gibbs state'. Recall the set $\mathcal{K}$ of cliques of the graph G.

(7.9) Definition. A probability measure π on Σ is called a *Gibbs state* if there exists a 'potential' function $\phi : 2^V \to \mathbb{R}$, satisfying $\phi_C = 0$ if $C \notin \mathcal{K}$, such that

$$\text{(7.10)} \qquad \pi(B) = \exp\left(\sum_{K\subseteq B} \phi_K\right), \qquad B \subseteq V.$$

We allow the empty set in the above summation, so that $\log \pi(\varnothing) = \phi_\varnothing$.

Gibbs states are thus named after Josiah Willard Gibbs, whose volume [87] made available the foundations of statistical mechanics. A simplistic motivation for the form of (7.10) is as follows. Suppose that each state σ has an energy E_σ, and a probability $\pi(\sigma)$. We constrain the average energy $E = \sum_\sigma E_\sigma \pi(\sigma)$ to be fixed, and we maximize the entropy

$$\eta(\pi) = -\sum_{\sigma\in\Sigma} \pi(\sigma) \log_2 \pi(\sigma).$$

With the aid of a Lagrange multiplier β, we find that

$$\pi(\sigma) \propto e^{-\beta E_\sigma}, \qquad \sigma \in \Sigma.$$

The theory of thermodynamics leads to the expression $\beta = 1/(kT)$ where k is Boltzmann's constant and T is (absolute) temperature. Formula (7.10) arises when the energy E_σ may be expressed as the sum of the energies of the sub-systems indexed by cliques.

(7.11) Theorem. *A positive probability measure π on Σ is a Markov field if and only if it is a Gibbs state. The potential function ϕ corresponding to the Markov field π is given by*

$$\phi_K = \sum_{L \subseteq K} (-1)^{|K \setminus L|} \log \pi(L), \qquad K \in \mathcal{K}.$$

A positive probability measure π is said to have the *local Markov property* if it satisfies the global property (7.8) for all *singleton* sets W and all $s \in \Sigma$. The global property evidently implies the local property, and it turns out that the two properties are equivalent. For notational convenience, we denote a singleton set $\{w\}$ as w.

(7.12) Proposition. *Let π be a positive probability measure on Σ. The following three statements are equivalent*:

(i) *π satisfies the global Markov property,*

(ii) *π satisfies the local Markov property,*

(iii) *for all $A \subseteq V$ and any pair $u, v \in V$ with $u \notin A$, $v \in A$ and $u \nsim v$,*

$$\frac{\pi(A \cup u)}{\pi(A)} = \frac{\pi(A \cup u \setminus v)}{\pi(A \setminus v)}. \tag{7.13}$$

Proof. First, assume (i), so that (ii) is implied trivially. Let $u \notin A$, $v \in A$, and $u \nsim v$. Applying (7.8) with $W = \{u\}$ and, for $w \neq u$, $s_w = 1$ if and only if $w \in A$, we find that

(7.14)

$$\begin{aligned}\frac{\pi(A \cup u)}{\pi(A) + \pi(A \cup u)} &= \pi(\sigma_u = 1 \mid \sigma_{V \setminus u} = A)\\ &= \pi(\sigma_u = 1 \mid \sigma_{\Delta u} = A \cap \Delta u)\\ &= \pi(\sigma_u = 1 \mid \sigma_{V \setminus u} = A \setminus v) \qquad \text{since } v \notin \Delta u\\ &= \frac{\pi(A \cup u \setminus v)}{\pi(A \setminus v) + \pi(A \cup u \setminus v)}.\end{aligned}$$

Equation (7.14) is equivalent to (7.13), whence (ii) and (iii) are equivalent under (i).

It remains to show that the local property implies the global property. The proof requires a short calculation, and may be done either by Theorem 7.1 or within the proof of Theorem 7.11. We follow the first route here. Assume that π is positive and satisfies the local Markov property. Then $u \perp v$ for all $u, v \in V$ with $u \nsim v$. By Theorem 7.1, there exist functions f_K, $K \in \mathcal{M}$, such that

$$\pi(A) = \prod_{K \in \mathcal{M}} f_K(A \cap K), \qquad A \subseteq V. \tag{7.15}$$

Let $W \subseteq V$. By (7.15), for $A \subseteq W$ and $C \subseteq V \setminus W$,

$$\pi(\sigma_W = A \mid \sigma_{V \setminus W} = C) = \frac{\prod_{K \in \mathcal{M}} f_K((A \cup C) \cap K)}{\sum_{B \subseteq W} \prod_{K \in \mathcal{M}} f_K((B \cup C) \cap K)}.$$

Any clique K with $K \cap W = \varnothing$ makes the same contribution $f_K(C \cap K)$ to both numerator and denominator, and may be cancelled. The remaining cliques are subsets of $\widehat{W} = W \cup \Delta W$, so that

$$\pi(\sigma_W = A \mid \sigma_{V \setminus W} = C) = \frac{\prod_{K \in \mathcal{M},\, K \subseteq \widehat{W}} f_K((A \cup C) \cap K)}{\sum_{B \subseteq W} \prod_{K \in \mathcal{M},\, K \subseteq \widehat{W}} f_K((B \cup C) \cap K)}.$$

The right side does not depend on $\sigma_{V \setminus \widehat{W}}$, whence

$$\pi(\sigma_W = A \mid \sigma_{V \setminus W} = C) = \pi(\sigma_W = A \mid \sigma_{\Delta W} = C \cap \Delta W)$$

as required for the global Markov property. □

Proof of Theorem 7.11. Assume first that π is a positive Markov field, and let

$$\phi_C = \sum_{L \subseteq C} (-1)^{|C \setminus L|} \log \pi(L), \qquad C \subseteq V. \tag{7.16}$$

By the inclusion–exclusion principle,

$$\log \pi(B) = \sum_{C \subseteq B} \phi_C, \qquad B \subseteq V,$$

and we need only show that $\phi_C = 0$ for $C \notin \mathcal{K}$. Suppose $u, v \in C$ and $u \nsim v$. By (7.16),

$$\phi_C = \sum_{L \subseteq C \setminus \{u,v\}} (-1)^{|C \setminus L|} \log \left(\frac{\pi(L \cup u \cup v)}{\pi(L \cup u)} \Big/ \frac{\pi(L \cup v)}{\pi(L)} \right),$$

which equals zero by the local Markov property and Proposition 7.12. Therefore, π is a Gibbs state with potential function ϕ.

Conversely, suppose that π is a Gibbs state with potential function ϕ. Evidently, π is positive. Let $A \subseteq V$, and $u \notin A$, $v \in A$ with $u \nsim v$. By (7.10),

$$\begin{aligned}
\log\left(\frac{\pi(A \cup u)}{\pi(A)}\right) &= \sum_{K \subseteq A \cup u,\, u \in K} \phi_K \\
&= \sum_{K \subseteq A \cup u \setminus v,\, u \in K} \phi_K \qquad \text{since } u \nsim v \text{ and } K \in \mathcal{K} \\
&= \log\left(\frac{\pi(A \cup u \setminus v)}{\pi(A \setminus v)}\right).
\end{aligned}$$

The claim follows by Proposition 7.12. □

We close this section with some notes on the history of Theorem 7.11. It may be derived from Brook's theorem, Theorem 7.1, but it is perhaps more informative to prove it directly as above via the inclusion–exclusion principle. It is normally attributed to Hammersley and Clifford, and it was circulated (with a more complicated formulation and proof) in an unpublished note of 1971, [115] (see [63]). Versions of the theorem may be found in the later work of several authors. The above proof is taken from [92], the author's earliest published paper and part of his 1972 MSc dissertation at Oxford University. The assumption of positivity is important, and complications arise for non-positive measures, see [166].

For applications of the Gibbs/Markov equivalence in statistics, see, for example, [142].

7.3 Ising and Potts models

In a famous experiment, a piece of iron is exposed to a magnetic field. The field is increased from zero to a maximum, and then diminished to zero. If the temperature is sufficiently low, the iron retains some residual magnetization, otherwise it does not. There is a critical temperature for this phenomenon, often named the *Curie point* after Pierre Curie, who reported this discovery in his 1895 thesis. The famous (Lenz–)Ising model for such ferromagnetism, [127], may be summarized as follows. Let particles be positioned at the points of some lattice in Euclidean space. Each particle may be in either of two states, representing the physical states of 'spin-up' and 'spin-down'. Spin-values are chosen at random according to a Gibbs state governed by interactions between neighbouring particles, and given in the following way.

Let $G = (V, E)$ be a finite graph representing part of the lattice. Each vertex $x \in V$ is considered as being occupied by a particle that has a random spin. Spins are assumed to come in two basic types ('up' and 'down'), and thus we take the set $\Sigma = \{-1, +1\}^V$ as the sample space. The appropriate probability mass function $\lambda_{\beta,J,h}$ on Σ has three parameters satisfying $\beta, J \in [0, \infty)$ and $h \in \mathbb{R}$, and is given by

$$\lambda_{\beta,J,h}(\sigma) = \frac{1}{Z_{\mathrm{I}}} e^{-\beta H(\sigma)}, \qquad \sigma \in \Sigma, \tag{7.17}$$

where the 'Hamiltonian' $H : \Sigma \to \mathbb{R}$ and the 'partition function' Z_{I} are given by

$$H(\sigma) = -J \sum_{e=\langle x,y\rangle \in E} \sigma_x \sigma_y - h \sum_{x \in V} \sigma_x, \qquad Z_{\mathrm{I}} = \sum_{\sigma \in \Sigma} e^{-\beta H(\sigma)}. \tag{7.18}$$

The physical interpretation of β is as the reciprocal $1/T$ of temperature, of J as the strength of interaction between neighbours, and of h as the external magnetic field. We shall consider here only the case of zero external-field, and we assume henceforth that $h = 0$.

Each edge has equal interaction strength J in the above formulation. Since β and J occur only as a product βJ, the measure $\lambda_{\beta,J,0}$ has effectively only a single parameter βJ. In a more complicated measure not studied here, different edges e are permitted to have different interaction strengths J_e. In the meantime we shall set $J = 1$, and write $\lambda_\beta = \lambda_{\beta,1,0}$

Whereas the Ising model permits only two possible spin-values at each vertex, the so-called (Domb–)Potts model [177] has a general number $q \ge 2$, and is governed by the following probability measure.

Let q be an integer satisfying $q \ge 2$, and take as sample space the set of vectors $\Sigma = \{1, 2, \dots, q\}^V$. Thus each vertex of G may be in any of q states. For an edge $e = \langle x, y\rangle$ and a configuration $\sigma = (\sigma_x : x \in V) \in \Sigma$, we write $\delta_e(\sigma) = \delta_{\sigma_x,\sigma_y}$ where $\delta_{i,j}$ is the Kronecker delta. The relevant probability measure is given by

$$\pi_{\beta,q}(\sigma) = \frac{1}{Z_{\mathrm{P}}} e^{-\beta H'(\sigma)}, \qquad \sigma \in \Sigma, \tag{7.19}$$

where $Z_{\mathrm{P}} = Z_{\mathrm{P}}(\beta, q)$ is the appropriate partition function (or normalizing constant) and the Hamiltonian H' is given by

$$H'(\sigma) = -\sum_{e=\langle x,y\rangle \in E} \delta_e(\sigma). \tag{7.20}$$

In the special case $q = 2$,

$$\delta_{\sigma_x,\sigma_y} = \tfrac{1}{2}(1 + \sigma_x\sigma_y), \quad \sigma_x, \sigma_y \in \{-1, +1\}, \tag{7.21}$$

It is easy to see in this case that the ensuing Potts model is simply the Ising model with an adjusted value of β, in that $\pi_{\beta,2}$ is the measure obtained from $\lambda_{\beta/2}$ by re-labelling the local states.

We mention one further generalization of the Ising model, namely the so-called n-vector or $\mathrm{O}(n)$ model. Let $n \in \{1, 2, \dots\}$ and let S^{n-1} be the set of vectors of

$\mathbb{R}^n$ with unit length, that is, the $(n-1)$-sphere. A 'model' is said to have $\mathrm{O}(n)$ symmetry if its Hamiltonian is invariant under the operation on S^{n-1} of $n \times n$ orthonormal matrices. One such model is the n-vector model on $G = (V, E)$, with Hamiltonian

$$H_n(\mathbf{s}) = -\sum_{e=\langle x,y\rangle \in E} \mathbf{s}_x \cdot \mathbf{s}_y, \qquad \mathbf{s} = (\mathbf{s}_v : v \in V) \in (S^{n-1})^V,$$

where $\mathbf{s}_x \cdot \mathbf{s}_y$ denotes the dot product. When $n = 1$, this is simply the Ising model. It is called the X/Y model when $n = 2$, and the Heisenberg model when $n = 3$.

The Ising and Potts models have very rich theories, and are amongst the most intensively studied of models of statistical mechanics. In 'classical' work, they are studied via cluster expansions and correlation inequalities. The so-called 'random-cluster model', developed by Fortuin and Kasteleyn around 1960, provides a single framework that incorporates the percolation, Ising, and Potts models, as well as electrical networks, uniform spanning trees and forests. It enables a representation of the two-point correlation function of a Potts model as a connection probability of an appropriate model of stochastic geometry, and this in turn allows the use of geometrical techniques already refined in the case of percolation. The random-cluster model is defined and described in Chapter 8, see also [98].

The $q = 2$ Potts model is of course the Ising model, and special features of the number 2 allow a special analysis for the Ising model not yet replicated for general Potts models. This method is termed the 'random-current representation', and it has been especially fruitful in the study of the phase transition of the Ising model on $\mathbb{L}^d$. See [3, 7, 10] and [98, Chap. 9].

7.4 Exercises

7.1. [166] Investigate the Gibbs/Markov equivalence for probability measures that have zeroes.

7.2. *Ising model.* Let $G = (V, E)$ be a finite graph, and let λ be the probability measure on $\Sigma = \{-1, +1\}^V$ given by

$$\lambda(\sigma) \propto \exp\left(\beta \sum_{e=\langle i,j\rangle} \sigma_i \sigma_j\right), \qquad \sigma \in \Sigma,$$

where $\beta > 0$. Thinking of Σ as a partially ordered set (where $\sigma \le \sigma'$ if and only if $\sigma_i \le \sigma_i'$ for all i), show that:

(a) for $v \in V$, $\lambda(\cdot \mid \sigma_v = -1) \le_{\mathrm{st}} \lambda \le_{\mathrm{st}} \lambda(\cdot \mid \sigma_v = +1)$,

(b) λ satisfies the FKG lattice condition, and hence is positively associated.

8

Random-Cluster Model

The basic properties of the model are summarized, and its relationship to the Ising and Potts models described. The phase transition is defined in terms of the infinite-volume measures. After an account of a number of areas meritorious of further research, there is a section devoted to planar duality and the conjectured value of the critical point on the square lattice. The random-cluster model is linked in more than one way to the study of a random even subgraph of a graph.

8.1 The random-cluster and Ising/Potts models

Let $G = (V, E)$ be a finite graph, and write $\Omega = \{0, 1\}^E$. For $\omega \in \Omega$, we write $\eta(\omega) = \{e \in E : \omega(e) = 1\}$ for the set of open edges, and $k(\omega)$ for the number of connected components[1], or 'clusters', of the subgraph $(V, \eta(\omega))$. The *random-cluster measure* on Ω, with parameters $p \in [0, 1]$, $q \in (0, \infty)$ is the probability measure given by

$$\phi_{p,q}(\omega) = \frac{1}{Z} \left\{ \prod_{e \in E} p^{\omega(e)} (1-p)^{1-\omega(e)} \right\} q^{k(\omega)}, \qquad \omega \in \Omega, \tag{8.1}$$

where $Z = Z_{G,p,q}$ is the normalizing constant.

Some history. This measure was introduced by Fortuin and Kasteleyn in a series of papers dated around 1970. They sought a unification of the theory of electrical networks, percolation, Ising, and Potts models, and were motivated by the observation that each of these systems satisfies a certain series/parallel law. Percolation is evidently retrieved by setting $q = 1$, and it turns out that electrical networks arise via the UST limit obtained on taking the limit $p, q \to 0$ in such a way that $q/p \to 0$. The relationship to Ising/Potts models is more interesting in that it involves a transformation of measures described next. In brief, connection probabilities for the random-cluster measure correspond to correlations

[1]It is important to include isolated vertices in this count.

for ferromagnetic Ising/Potts models, and this allows a geometrical interpretation of their correlation structure.

A fuller account of the random-cluster model and its history and associations may be found in [98]. When the emphasis is upon its connection to Ising/Potts models, the random-cluster model is often called the 'FK representation'.

In the remainder of this section, we summarize the relationship between a Potts model on $G = (V, E)$ with an integer number q of local states, and the random-cluster measure $\phi_{p,q}$. As configuration space for the Potts model, we take $\Sigma = \{1, 2, \dots, q\}^V$. Let F be the subset of the product space $\Sigma \times \Omega$ containing all pairs (σ, ω) such that: for every edge $e = \langle x, y\rangle \in E$, if $\omega(e) = 1$ then $\sigma_x = \sigma_y$. That is, F contains all pairs (σ, ω) such that σ is constant on each cluster of ω.

Let $\phi_p = \phi_{p,1}$ be product measure on ω with density p, and let μ be the probability measure on $\Sigma \times \Omega$ given by

$$\mu(\sigma, \omega) \propto \phi_p(\omega) 1_F(\sigma, \omega), \qquad (\sigma, \omega) \in \Sigma \times \Omega, \tag{8.2}$$

where 1_F is the indicator function of F.

Four calculations are now required, in order to determine the two marginal measures of μ and the two conditional measures. It turns out that the two marginals are exactly the q-state Potts measure on Σ (with suitable pair-interaction) and the random-cluster measure $\phi_{p,q}$.

Marginal on Σ. When we sum $\mu(\sigma, \omega)$ over $\omega \in \Omega$, we have a free choice except in that $\omega(e) = 0$ whenever $\sigma_x \neq \sigma_y$. That is, if $\sigma_x = \sigma_y$, there is no constraint on the local state $\omega(e)$ of the edge $e = \langle x, y\rangle$; the sum for this edge is simply $p + (1 - p) = 1$. We are left with edges e with $\sigma_x \neq \sigma_y$, and therefore

$$\mu(\sigma, \cdot) := \sum_{\omega \in \Omega} \mu(\sigma, \omega) \propto \prod_{e \in E} (1 - p)^{1 - \delta_e(\sigma)}, \tag{8.3}$$

where $\delta_e(\sigma)$ is the Kronecker delta

$$\delta_e(\sigma) = \delta_{\sigma_x, \sigma_y} \qquad e = \langle x, y\rangle \in E. \tag{8.4}$$

Otherwise expressed,

$$\mu(\sigma, \cdot) \propto \exp\left\{\beta \sum_{e \in E} \delta_e(\sigma)\right\}, \qquad \sigma \in \Sigma,$$

where

$$p = 1 - e^{-\beta}. \tag{8.5}$$

This is the Potts measure $\pi_{\beta,q}$ of (7.19). Note that $\beta \geq 0$, which is to say that the model is ferromagnetic.

Marginal on Ω. For given ω, the constraint on σ is that it be constant on open clusters. There are $q^{k(\omega)}$ such spin configurations, and $\mu(\sigma, \omega)$ is constant on this set. Therefore,

$$\begin{aligned}\mu(\cdot, \omega) := \sum_{\sigma \in \Sigma} \mu(\sigma, \omega) &\propto \left\{\prod_{e \in E} p^{\omega(e)}(1-p)^{1-\omega(e)}\right\} q^{k(\omega)} \\ &\propto \phi_{p,q}(\omega), \qquad \omega \in \Omega.\end{aligned}$$

The conditional measures. Given ω, the conditional measure on Σ is obtained by putting (uniformly) random spins on entire clusters of ω, constant on given clusters, and independent between clusters. Given σ, the conditional measure on Ω is obtained by setting $\omega(e) = 0$ if $\delta_e(\sigma) = 0$, and otherwise $\omega(e) = 1$ with probability p (independently of other edges).

The 'two-point correlation function' of the Potts measure $\pi_{\beta,q}$ on $G = (V, E)$ is the function $\tau_{\beta,q}$ given by

$$\tau_{\beta,q}(x, y) = \pi_{\beta,q}(\sigma_x = \sigma_y) - \frac{1}{q}, \qquad x, y \in V.$$

The 'two-point connectivity function' of the random-cluster measure $\phi_{p,q}$ is the probability $\phi_{p,q}(x \leftrightarrow y)$ of an open path from x to y. It turns out that these 'two-point functions' are (except for a constant factor) the same.

(8.6) Theorem [133]. *For* $q \in \{2, 3, \dots\}$, $\beta \geq 0$, *and* $p = 1 - e^{-\beta}$,

$$\tau_{\beta,q}(x, y) = (1 - q^{-1})\phi_{p,q}(x \leftrightarrow y).$$

Proof. We work with the conditional measure $\mu(\sigma \mid \omega)$ thus:

$$\begin{aligned}\tau_{\beta,q}(x, y) &= \sum_{\sigma,\omega} \big[1_{\{\sigma_x = \sigma_y\}}(\sigma) - q^{-1}\big]\mu(\sigma, \omega) \\ &= \sum_{\omega} \phi_{p,q}(\omega) \sum_{\sigma} \mu(\sigma \mid \omega)\big[1_{\{\sigma_x = \sigma_y\}}(\sigma) - q^{-1}\big] \\ &= \sum_{\omega} \phi_{p,q}(\omega)\big[(1 - q^{-1})1_{\{x \leftrightarrow y\}}(\omega) + 0 \cdot 1_{\{x \not\leftrightarrow y\}}(\omega)\big] \\ &= (1 - q^{-1})\phi_{p,q}(x \leftrightarrow y),\end{aligned}$$

and the claim is proved. □

8.2 Basic properties

We list some of the fundamental properties of random-cluster measures in this section.

(8.7) Theorem. *The measure $\phi_{p,q}$ satisfies the FKG lattice condition if $q \geq 1$, and is thus positively associated.*

Proof. If $p = 0, 1$, the conclusion is obvious. Assume $0 < p < 1$, and check the FKG lattice condition (4.12), which amounts to the assertion that

$$k(\omega \vee \omega') + k(\omega \wedge \omega') \geq k(\omega) + k(\omega'), \qquad \omega, \omega' \in \Omega.$$

This is left as a graph-theoretic exercise for the reader. □

(8.8) Theorem. Comparison inequalities [81]. *We have that*

$$\phi_{p',q'} \leq_{\mathrm{st}} \phi_{p,q} \quad \text{if} \quad p' \leq p,\ q' \geq q,\ q' \geq 1, \tag{8.9}$$

$$\phi_{p',q'} \geq_{\mathrm{st}} \phi_{p,q} \quad \text{if} \quad \frac{p'}{q'(1-p')} \geq \frac{p}{q(1-p)},\ q' \geq q,\ q' \geq 1. \tag{8.10}$$

Proof. This follows by the Holley inequality, Theorem 4.4, on checking condition (4.5). □

In the next theorem, the role of the graph G is emphasized in the use of the notation $\phi_{G,p,q}$. The graph $G\backslash e$ (respectively, $G.e$) is obtained from G by deleting (respectively, contracting) the edge e.

(8.11) Theorem [81]. *Let $e \in E$.*

(a) *Conditional on $\omega(e) = 0$, the measure obtained from $\phi_{G,p,q}$ is $\phi_{G\backslash e,p,q}$.*
(b) *Conditional on $\omega(e) = 1$, the measure obtained from $\phi_{G,p,q}$ is $\phi_{G.e,p,q}$.*

Proof. This is an elementary calculation of conditional probabilities. □

Maybe mention UST, and negative association/disjoint occurrence.

8.3 Infinite-volume limits and phase transition

Let $d \geq 2$, and $\Omega = \{0,1\}^{\mathbb{E}^d}$. The appropriate σ-field of Ω is the σ-field $\mathcal{F}$ generated by the finite-dimensional sets. Let Λ be a finite box in $\mathbb{Z}^d$. For $b \in \{0,1\}$ define

$$\Omega_\Lambda^b = \{\omega \in \Omega : \omega(e) = b \text{ for } e \notin \mathbb{E}_\Lambda\},$$

where $\mathbb{E}_A$ is the set of edges of $\mathbb{L}^d$ joining pairs of vertices belonging to A. On Ω_Λ^b we define a random-cluster measure $\phi_{\Lambda,p,q}^b$ as follows. For $p \in [0,1]$ and $q \in (0,\infty)$, let

$$\text{(8.12)} \quad \phi_{\Lambda,p,q}^b(\omega) = \frac{1}{Z_{\Lambda,p,q}^b}\left\{\prod_{e\in\mathbb{E}_\Lambda} p^{\omega(e)}(1-p)^{1-\omega(e)}\right\} q^{k(\omega,\Lambda)}, \qquad \omega \in \Omega_\Lambda^b,$$

where $k(\omega,\Lambda)$ is the number of clusters of $(\mathbb{Z}^d, \eta(\omega))$ that intersect Λ (here, as before, $\eta(\omega) = \{e \in \mathbb{E}^d : \omega(e) = 1\}$ is the set of open edges). The boundary condition $b = 0$ (respectively, $b = 1$) is sometimes termed 'free' (respectively, 'wired').

(8.13) Theorem [93]. *Let $q \geq 1$. The weak limits*

$$\phi_{p,q}^b = \lim_{\Lambda\to\mathbb{Z}^d} \phi_{\Lambda,p,q}^b, \qquad b = 0, 1,$$

exist, and are translation-invariant and ergodic.

Proof. Let A be an increasing cylinder event defined in terms of the edges lying in some finite set S. If $\Lambda \subseteq \Lambda'$ and Λ includes the 'base' S of the cylinder A,

$$\phi_{\Lambda,p,q}^1(A) = \phi_{\Lambda',p,q}^1(A \mid \text{all edges in } \mathbb{E}_{\Lambda'\setminus\Lambda} \text{ are open}) \geq \phi_{\Lambda',p,q}^1(A),$$

where we have used Theorem 8.11 and the FKG inequality. Therefore, the limit $\lim_{\Lambda\to\mathbb{Z}^d} \phi_{\Lambda,p,q}^1(A)$ exists by monotonicity. Since $\mathcal{F}$ is generated by such events A, the weak limit $\phi_{p,q}^1$ exists. A similar argument is valid in the case $b = 0$.

Translation-invariance holds in very much the same way as in the proof of Theorem 2.10. The proof of ergodicity is deferred to Exercises 8.9–8.10. □

The measures $\phi_{p,q}^0$ and $\phi_{p,q}^1$ are called 'random-cluster measures' on $\mathbb{L}^d$ with parameters p and q, and they are extremal in the following sense. One may generate ostensibly larger families of infinite-volume random-cluster measures by either of two routes. In the first, one considers measures $\phi_{\Lambda,p,q}^\xi$ on $\mathbb{E}_\Lambda$ with more general boundary conditions ξ, in order to construct a set $\mathcal{W}_{p,q}$ of 'weak-limit random-cluster measures'. The second uses a type of Dobrushin–Lanford–Ruelle (DLR) formalism rather than weak limits (see [93] and [98, Chap. 4]). More precisely, one considers measures μ on $(\Omega, \mathcal{F})$ whose measure on any box Λ, conditional on the state ξ off Λ, is the conditional random-cluster measure $\phi_{\Lambda,p,q}^\xi$. Such a μ is called a 'DLR random-cluster measure', and we write $\mathcal{R}_{p,q}$ for the set of DLR measures. The relationship between $\mathcal{W}_{p,q}$ and $\mathcal{R}_{p,q}$ is not fully understood, and we make one remark about this. Any element μ of the closed convex hull of $\mathcal{W}_{p,q}$ with the so-called '0/1-infinite-cluster property' (that is, $\mu(I \in \{0,1\}) = 1$ where I is the number of infinite open clusters) belongs to $\mathcal{R}_{p,q}$, see [98, Sect. 4.4]. The standard way of showing the 0/1-infinite-cluster property is via the Burton–Keane argument used in the proof of Theorem 5.22. One may show, in particular, that $\phi^0_{p,q}, \phi^1_{p,q} \in \mathcal{R}_{p,q}$.

Henceforth we assume that $q \ge 1$. The measures $\phi^0_{p,q}$ and $\phi^1_{p,q}$ are extremal in the sense that

(8.14) $$\phi^0_{p,q} \le_{\text{st}} \phi_{p,q} \le_{\text{st}} \phi^1_{p,q}, \qquad \phi_{p,q} \in \mathcal{W}_{p,q} \cup \mathcal{R}_{p,q},$$

whence there exists a unique random-cluster measure (in either of the above senses) if and only if $\phi^0_{p,q} = \phi^1_{p,q}$. It is a general fact that such extremal measures are invariably ergodic, see [86, 98].

Turning to the question of phase transition, and remembering percolation, we define the *percolation probabilities*

(8.15) $$\theta^b(p,q) = \phi^b_{p,q}(0 \leftrightarrow \infty), \qquad b = 0, 1,$$

that is, the probability that 0 belongs to an infinite open cluster. The corresponding *critical values* are given by

(8.16) $$p^b_{\text{c}}(q) = \sup\{p : \theta^b(p,q) = 0\}, \qquad b = 0, 1.$$

Faced possibly with two (or more) distinct critical values, we present the following result.

(8.17) Theorem [9, 93]. *Let $d \ge 2$ and $q \ge 1$. We have that*:

(i) $\phi^0_{p,q} = \phi^1_{p,q}$ *if* $\theta^1(p,q) = 0$,

(ii) *there exists a countable subset* $\mathcal{D}_{d,q}$ *of* $[0,1]$, *possibly empty, such that* $\phi^0_{p,q} = \phi^1_{p,q}$ *if and only if* $p \notin \mathcal{D}_{d,q}$.

Sketch proof. The argument for (i) is as follows. Clearly,

(8.18) $$\theta^1(p,q) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \phi^1_{p,q}(0 \leftrightarrow \partial\Lambda).$$

Suppose $\theta^1(p,q) = 0$, and consider a large box Λ with 0 in its interior. On building the clusters that intersect the boundary $\partial\Lambda$, with high probability we do not reach 0. That is, with high probability, there exists a 'cut-surface' S between 0 and $\partial\Lambda$ that comprises only closed edges. The position of S may be taken to be measurable on its exterior, whence the conditional measure on the interior of S is a free random-cluster measure. Passing to the limit as $\Lambda \uparrow \mathbb{Z}^d$, we find that the free and wired measures are equal.

The argument for (ii) is based on a classical method of statistical mechanics using convexity. Let $Z_{G,p,q}$ be the partition function of the random-cluster model on a graph $G = (V, E)$, and set

$$Y_{G,p,q} = (1-p)^{-|E|} Z_{G,p,q} = \sum_{\omega \in \{0,1\}^E} e^{\pi|\eta(\omega)|} q^{k(\omega)},$$

where $\pi = \log[p/(1-p)]$. It is easily seen that $\log Y_{G,p,q}$ is a convex function of π. By a standard method based on the negligibility of the boundary of a large box Λ compared with its volume, the limit 'pressure function'

$$\Pi(\pi, q) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \left\{ \frac{1}{|\mathbb{E}_\Lambda|} \log Y^\xi_{\Lambda,p,q} \right\}$$

exists and is independent of the boundary configuration $\xi \in \Omega$. Since Π is the limit of convex functions of π, it is convex, and hence differentiable except on some countable set $\mathcal{D}$ of values of π. Furthermore, for $\pi \notin \mathcal{D}$, the derivative of $|\mathbb{E}_\Lambda|^{-1} \log Y^\xi_{\Lambda,p,q}$ converges to that of Π. The former derivative may be interpreted in terms of the edge-density of the measures, and therefore the limits of the last are independent of ξ for any π at which $\Pi(\pi, q)$ is differentiable.[2] Uniqueness of random-cluster measures follows by (8.14) and stochastic ordering: if μ_1, μ_2 are probability measures on $(\Omega, \mathcal{F})$ with $\mu_1 \le_{\text{st}} \mu_2$ and satisfying

$$\mu_1(e \text{ is open}) = \mu_2(e \text{ is open}), \qquad e \in \mathbb{E},$$

then[3] $\mu_1 = \mu_2$. □

By Theorem 8.17, $\theta^0(p, q) = \theta^1(p, q)$ for $p \notin \mathcal{D}_{d,q}$, whence $p_c^0(q) = p_c^1(q)$. Henceforth we refer to the critical value as $p_c = p_c(q)$. The following is an important conjecture.

(8.19) Conjecture. *There exists $Q = Q(d)$ such that*:

(i) *if $q < Q(d)$, then $\theta^1(p_c, q) = 0$ and $\mathcal{D}_{d,q} = \varnothing$,*

(ii) *if $q > Q(d)$, then $\theta^1(p_c, q) > 0$ and $\mathcal{D}_{d,q} = \{p_c\}$.*

Reference physics literature for $q \ge 4$, $d = 2$.

In the physical vernacular, there is conjectured a critical value of q beneath which the phase transition is continuous ('second order') and above which it is discontinuous ('first order'). Following work of Roman Kotecký and Senya Shlosman [139], it was proved in [140] that there is a first-order transition for large q, see [98, Sects 6.4, 7.5]. It is expected that

$$Q(d) = \begin{cases} 4 & \text{if } d = 2, \\ 2 & \text{if } d \ge 6. \end{cases}$$

This may be contrasted with the best current estimate in two dimensions, namely $Q(2) \le 25.72$, see [98, Sect. 6.4].

It is a basic fact that $p_c(q)$ is non-trivial.

[2] Expand

[3] *Exercise.* Recall Strassen's Theorem 4.2.

(8.20) Theorem [9]. *If $d \geq 2$ and $q \geq 1$ then $0 < p_c(q) < 1$.*

It is an open problem to find a satisfactory definition of $p_c(q)$ for $q < 1$, although it may be shown by the comparison inequalities (Theorem 8.8) that there is no infinite cluster for $q \in (0, 1)$ and small p, and conversely there is an infinite cluster for $q \in (0, 1)$ and large p.

Proof. Let $q \geq 1$. By Theorem 8.8, $\phi^1_{p',1} \leq_{\text{st}} \phi^1_{p,q} \leq_{\text{st}} \phi_{p,1}$, where $p' = p/[p + q(1-p)]$. We apply this inequality to the increasing event $\{0 \leftrightarrow \partial\Lambda\}$, and let $\Lambda \uparrow \mathbb{Z}^d$ to obtain via (8.22) that

$$p_c(1) \leq p_c(q) \leq \frac{q p_c(1)}{1 + (q-1)p_c(1)}, \qquad q \geq 1, \tag{8.21}$$

where $0 < p_c(1) < 1$ by Theorem 3.2. □

Finally, we review the relationship between the random-cluster and Potts phase transitions. The 'order parameter' of the Potts model is the 'magnetization' given by

$$M(\beta, q) = \lim_{\Lambda \to \mathbb{Z}^d} \left\{ \pi^1_{\Lambda,\beta}(\sigma_0 = 1) - \frac{1}{q} \right\},$$

where $\pi^1_{\Lambda,\beta}$ is the Potts measure on Λ 'with boundary condition 1'. We may think of $M(\beta, q)$ as a measure of the degree to which the boundary condition '1' is noticed at the origin after taking the infinite-volume limit. By an application of Theorem 8.6 to a suitable graph obtained from Λ,

$$\pi^1_{\Lambda,q}(\sigma_0 = 1) - \frac{1}{q} = (1 - q^{-1})\phi^1_{\Lambda,p,q}(0 \leftrightarrow \partial\Lambda)$$

where $p = 1 - e^{-\beta}$. It may be deduced[4] that

$$\theta^1(p, q) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \phi^1_{\Lambda,p,q}(0 \leftrightarrow \partial\Lambda). \tag{8.22}$$

Therefore

$$\begin{aligned} M(\beta, q) &= (1 - q^{-1}) \lim_{\Lambda \to \mathbb{Z}^d} \phi^1_{\Lambda,p,q}(0 \leftrightarrow \partial\Lambda) \\ &= (1 - q^{-1})\theta^1(p, q), \end{aligned}$$

by (8.22). That is, $M(\beta, q)$ and $\theta^1(p, q)$ differ by the factor $1 - q^{-1}$.

[4] Exercise 8.8.

8.4 Open problems

Many questions remain at least partly unanswered for the random-cluster model, and we list a few of these here. Further details may be found in [98].

A. *The case* $q < 1$. Less is known when $q < 1$ owing to the failure of the FKG inequality. A possibly optimistic conjecture is that some version of negative association holds when $q < 1$, and this might imply the existence of infinite-volume limits. One may use comparison arguments to study infinite-volume random-cluster measures for sufficiently small or large p, but there is no proof of the existence of a unique point of phase transition. A certain amount is known in the limit as $q \to 0$, depending on how p behaves in this limit.

BK?

The case $q < 1$ is of more mathematical than physical interest, although the various limits as $q \to 0$ are relevant to the theory of algorithms and complexity.

Henceforth, we assume $q \ge 1$.

B. *Exponential decay.* Prove that

$$\phi_{p,q}\big(0 \leftrightarrow \partial[-n,n]^d\big) \le e^{-\alpha n}, \qquad n \ge 1,$$

for some $\alpha = \alpha(p,q)$ satisfying $\alpha > 0$ when $p < p_c(q)$. This has been proved for sufficiently small values of p, but no proof is known (for general q and any given $d \ge 2$) right up to the critical point.

The case $q = 2$ is special, since this corresponds to the Ising model, for which the random-current representation has allowed a rich theory, see [98, Sect. 9.3]. Exponential decay is proved to hold for general d, when $q = 2$, and also for sufficiently large q (see D below).

C. *Uniqueness of random-cluster measures.* Prove all or part of Conjecture 8.19. That is, show that $\phi_{p,q}^0 = \phi_{p,q}^1$ for $p \ne p_c(q)$. And, furthermore, that uniqueness holds when $p = p_c(q)$ if and only if q is sufficiently small.

These statements are trivial when $q = 1$, and uniqueness is proved when $q = 2$ and $p \ne p_c(2)$, using the theory of the Ising model alluded to above. The situation is curious when $q = 2$ and $p = p_c(2)$, in that uniqueness is proved so long as $d \ne 3$, see [98, Sect. 9.4].

When q is sufficiently large, it is known as in D below that there is a unique random-cluster measure when $p \ne p_c(q)$ and a multiplicity of such measures when $p = p_c(q)$.

D. *First/second order phase transition.* Much interest in Potts and random-cluster measures is focussed on the fact that nature of the phase transition depends on whether q is small or large, see for example Conjecture 8.19. For small q, the singularity is expected to be continuous and of power type. For large q, there is a discontinuity in the order parameter $\theta^1(\cdot, q)$, and a 'mass gap' at the critical point (that is, when $p = p_c(q)$, the $\phi_{p,q}^0$-probability of a long path decays exponentially, while the $\phi_{p,q}^1$-probability is bounded away from 0).

Of the possible questions, we ask for a proof of the existence of a value $Q = Q(d)$ separating the second- from the first-order transition.

E. *Slab critical point.* It was important for supercritical percolation in three and more dimensions to show that percolation in $\mathbb{L}^d$ implies percolation in a sufficiently fat 'slab', see Theorem 5.17. A version of the corresponding problem for the random-cluster model is as follows. Let $q \geq 1$ and $d \geq 3$, and write $S(L, n)$ for the 'slab'

$$S(L, n) = [0, L-1] \times [-n, n]^{d-1}.$$

Let $\psi_{L,n,p,q} = \phi^0_{S(L,n),p,q}$ be the random-cluster measure on $S(L, n)$ with parameters p, q, and free boundary conditions. Write $\Pi(p, L)$ for the property that:

$$\exists \alpha > 0 \text{ such that: } \forall n \geq 1,\ \forall x \in S(L, n),\ \psi_{L,n,p,q}(0 \leftrightarrow x) > \alpha.$$

It is not hard to see that $\Pi(p, L) \Rightarrow \Pi(p', L')$ if $p \leq p'$ and $L \leq L'$, and it is thus natural to define

$$(8.23) \qquad \widehat{p}_c(q, L) = \inf\big\{p : \Pi(p, L) \text{ occurs}\big\}, \quad \widehat{p}_c(q) = \lim_{L\to\infty} \widehat{p}_c(q, L).$$

[5] Clearly, $p_c(q) \leq \widehat{p}_c(q) < 1$. It is believed that equality holds in that $\widehat{p}_c(q) = p_c(q)$, and it is a major open problem to prove this. A positive resolution would have implications for the exponential decay of truncated cluster-sizes, and for the existence of a Wulff crystal for all $p > p_c(q)$ and $q \geq 1$. See Figure 5.3 and [55, 56, 57].

F. *Roughening transition.* While it is believed that there is a *unique* random-cluster measure except possibly at the critical point, there can exist a multitude of random-cluster-type measures with the striking property of non-translation-invariance. Take a box $\Lambda_n = [-n, n]^d$ in $d \geq 3$ dimensions (the following construction fails in 2 dimensions). We may think of $\partial\Lambda_n$ as comprising a northern and southern hemisphere, with the 'equator' $\{x \in \partial\Lambda_n : x_d = 0\}$ as interface. Let $\overline{\phi}_{n,p,q}$ be the random-cluster measure on Λ_n with a wired boundary condition on the northern (respectively, southern) hemisphere and conditioned on the event that no open path joins a point of the northern to a point of the southern hemisphere. By the compactness of Ω, the sequence $(\overline{\phi}_{n,p,q} : n \geq 1)$ possesses weak limits. Let $\overline{\phi}_{p,q}$ be such a weak limit.

It is a geometrical fact that, in any configuration ω on Λ_n, there exists an interface $I(\omega)$ separating the points joined to the northern hemisphere from those joined to the southern hemisphere. This interface passes around the equator, and its closest point to the origin is at some distance H_n, say. It may be shown that, for $q \geq 1$ and sufficiently large p, the laws of the H_n are tight, whence the weak limit

[5] Use FKG, explain

$\overline{\phi}_{p,q}$ is not translation-invariant. Such measures are termed 'Dobrushin measures' after their discovery for the Ising model in [64].

There remain two important questions. Firstly, for $d \geq 3$ and $q \geq 1$, does there exist a value $\widetilde{p}(q)$ such that Dobrushin measures exist for $p > \widetilde{p}(q)$ and not for $p < \widetilde{p}(q)$? And secondly, for what dimensions d do Dobrushin measures exist for all $p > p_c(q)$? A fuller account may be found in [98, Chap. 7].

G. *In two dimensions.* There remain some intriguing conjectures in the playground of the square lattice $\mathbb{L}^2$, and some of these are described in the next section.

8.5 In two dimensions

Consider the special case of the square lattice $\mathbb{L}^2$. Random-cluster measures on $\mathbb{L}^2$ have a property of self-duality that generalizes that of bond percolation. (We recall the discussion of duality after equation (3.7).) The most provocative conjecture is that the critical point equals the so-called self-dual point.

(8.24) Conjecture. *For $d = 2$ and $q \geq 1$,*

$$p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}}.$$

This formula is proved rigorously when $q = 1$ (percolation), when $q = 2$ (Ising model), and for sufficiently large values of q (namely $q \geq 25.72$).

The conjecture is motivated as follows. Let $G = (V, E)$ be a finite planar graph, and $G_d = (V_d, E_d)$ its dual graph. To each $\omega \in \Omega = \{0,1\}^E$, there corresponds the dual configuration $\omega_d \in \Omega_d = \{0,1\}^{E_d}$, given by

$$\omega_d(e_d) = 1 - \omega(e), \qquad e \in E.$$

(Note that this definition of the dual configuration differs from that used in Chapter 3 for percolation.) By drawing a picture, one may become convinced that every face of $(V, \eta(\omega))$ contains a unique component of $(V_d, \eta(\omega_d))$, and therefore the number $f(\omega)$ of faces (including the infinite face) of $(V, \eta(\omega))$ satisfies

$$f(\omega) = k(\omega_d). \tag{8.25}$$

The random-cluster measure on G satisfies

$$\phi_{G,p,q}(\omega) \propto \left(\frac{p}{1-p}\right)^{|\eta(\omega)|} q^{k(\omega)}.$$

Using (8.25), Euler's formula,

$$k(\omega) = |V| - |\eta(\omega)| + f(\omega) - 1, \tag{8.26}$$

and the fact that $|\eta(\omega)| + |\eta(\omega_d)| = |E|$, we have that

$$\phi_{G,p,q}(\omega) \propto \left(\frac{q(1-p)}{p}\right)^{|\eta(\omega_d)|} q^{k(\omega_d)},$$

which is to say that

$$\phi_{G,p,q}(\omega) = \phi_{G_d,p_d,q}(\omega_d), \qquad \omega \in \Omega, \tag{8.27}$$

where

$$\frac{p_d}{1-p_d} = \frac{q(1-p)}{p}. \tag{8.28}$$

The unique fixed point of the mapping $p \mapsto p_d$ is given by $p = \kappa_q$ where κ is the 'self-dual point'

$$\kappa_q = \frac{\sqrt{q}}{1+\sqrt{q}}.$$

Turning to the square lattice, let $G = \Lambda = [0,n]^2$, with dual graph $G_d = \Lambda_d$ obtained from the box $[-1,n]^2 + (\frac{1}{2},\frac{1}{2})$ by identifying all boundary vertices. By (8.27),

$$\phi^0_{\Lambda,p,q}(\omega) = \phi^1_{\Lambda_d,p_d,q}(\omega_d) \tag{8.29}$$

for configurations ω on Λ (and with a small 'fix' on the boundary of Λ_d). Letting $n \to \infty$, we obtain that

$$\phi^0_{p,q}(A) = \phi^1_{p_d,q}(A_d) \tag{8.30}$$

for all cylinder events A, where $A_d = \{\omega_d : \omega \in A\}$.

The duality relation (8.30) is useful, especially if $p = p_d = \kappa_q$. In particular, the proof that $\theta(\frac{1}{2}) = 0$ for percolation (see Theorem 5.33) may be adapted to obtain

$$\theta^0(\kappa_q, q) = 0, \tag{8.31}$$

whence

$$p_c(q) \ge \frac{\sqrt{q}}{1+\sqrt{q}}, \qquad q \ge 1. \tag{8.32}$$

In order to obtain the formula of Conjecture 8.24, it would be enough to show that,

$$\phi^0_{p,q}\big(0 \leftrightarrow \partial[-n,n]^2\big) \le \frac{A}{n}, \qquad n \ge 1,$$

where $A = A(p,q) < \infty$ for $p < \kappa_q$. See [89, 98].

The case $q = 2$ is very special, because it is related to the Ising model, for which there is a rich and exact theory going back to Onsager [168]. As an illustration of this connection in action, we include a proof that the wired random-cluster measure has no infinite cluster at the self-dual point. The corresponding conclusion in believed to hold if and only if $q \le 4$, but a full proof is elusive.

(8.33) Theorem. *For $d = 2$, $\theta^1(\kappa_2, 2) = 0$.*

Proof. Of the several proofs of this statement, we summarise the recent simple proof of Werner [210]. Let $q = 2$, and write $\phi^b = \phi^b_{p_{\mathrm{sd}}(q),q}$.

Let $\omega \in \Omega$ be a configuration of the random-cluster model sampled according to ϕ^0. To each open cluster of ω we allocate the spin $+1$ with probability $\frac{1}{2}$, and -1 otherwise. Thus, spins are constant within clusters, and independent between clusters. Let σ be the resulting spin configuration, and let μ^0 be its law. We do the same with ω sampled from ϕ^1, with the difference that any infinite cluster is allocated the spin $+1$. It is not hard to see[6] that the resulting measure μ^1 is the infinite-volume Ising measure with boundary condition $+1$. The spin-space $\Sigma = \{-1, +1\}^{\mathbb{Z}^2}$ is a partially ordered set, and it may be checked using the Holley inequality[7], Theorem 4.4, and passing to an infinite-volume limit that

$$\mu^0 \leq_{\mathrm{st}} \mu^1. \tag{8.34}$$

We shall be interested in two notions of connectivity in $\mathbb{Z}^2$, the first of which is the usual one, denoted $\leftrightarrow$. If we add both diagonals to each face of $\mathbb{Z}^2$, we obtain a new graph with so-called $*$-connectivity relation denoted $\leftrightarrow_*$. A cycle in this new graph is called a $*$-cycle.

Each $\sigma \in \Sigma$ amounts to a partition of $\mathbb{Z}^2$ into maximal clusters with constant spin. A cluster labelled $+1$ (respectively, -1) is called a $(+)$-cluster (respectively, $(-)$-cluster). Let $N^+(\sigma)$ (respectively, $N^-(\sigma)$) be the number of infinite $(+)$-clusters (respectively, $(-)$-clusters).

By (8.31), $\phi^0(0 \leftrightarrow \infty) = 0$, whence, by Exercise 8.14, μ^0 is ergodic. One may apply the Burton–Keane argument of Section 5.3 to deduce that:

$$\text{either} \quad \mu^0(N^+ = 1) = 1 \quad \text{or} \quad \mu^0(N^+ = 0) = 1.$$

One may now use Zhang's argument (as in the proof of (8.31) and Theorem 5.33), and the fact that N^+ and N^- have the same law, to deduce that

$$\mu^0(N^+ = 0) = \mu^0(N^- = 0) = 1. \tag{8.35}$$

Let A be an increasing cylinder event of Σ defined in terms of states of vertices in some box Λ. By (8.35), there are (μ^0-a.s.) no infinite $(-)$-clusters intersecting Λ, so that Λ lies in the interior of some $*$-cycle labelled $+1$. let $\Lambda_n = [-n, n]^2$ with n large, and let D_n be the event that Λ_n contains a $*$-cycle labelled $+1$ with Λ in its interior. By the above, $\mu^0(D_n) \to 1$ as $n \to \infty$. The event D_n is an increasing subset of Σ, whence, by (8.34),

$$\mu^1(D_n) \to 1 \qquad \text{as } n \to \infty.$$

[6] This is formalized in [98, Sect. 4.6]; see also Exercise 8.14.
[7] See Exercise 7.2.

On D_n, we find the 'outermost' $*$-cycle H labelled $+1$; this may be constructed explicitly via the boundaries of the $(-)$-clusters intersecting $\partial\Lambda_n$. Since H is outermost, the conditional measure of μ^1 (given D_n), restricted to Λ, is stochastically smaller than μ^0. On letting $n \to \infty$, we obtain $\mu^1(A) \le \mu^0(A)$, which is to say that $\mu^1 \le_{\mathrm{st}} \mu^0$. By (8.34), $\mu^0 = \mu^1$.

By (8.35), $\mu^1(N^+ = 0) = 1$, so that $\theta^1(\kappa_2, 2) = 0$ as claimed. □

Last, but definitely not least, we turn towards SLE and random-cluster/Ising models. Stanislav Smirnov has recently proved the convergence of re-scaled boundaries of large clusters of the critical random-cluster model on $\mathbb{L}^2$ to $\mathrm{SLE}_{16/3}$. The corresponding critical Ising model has spin-cluster boundaries converging to SLE_3. These results are having major impact on our understanding of the Ising model.

This[8] section ends with two open problems concerning exponential decay and/or SLE. Each Ising spin-configuration $\sigma \in \{-1, +1\}^V$ on a graph $G = (V, E)$ gives rise to a subgraph $G^\sigma = (V, E^\sigma)$ of G where

$$E^\sigma = \{e = \langle x, y\rangle \in E : \sigma_x = \sigma_y\}. \tag{8.36}$$

If G is planar, the boundary of any connected component of G^σ corresponds to a cycle in the dual graph G_{d}, and the union of all such cycles is a (random) even subgraph of G_{d} (see the next section).

We shall consider the Ising model on the square and triangular lattices, with inverse-temperature β satisfying $0 \le \beta \le \beta_{\mathrm{c}}$, where β_{c} is the critical value. By (8.5),

$$e^{-2\beta_{\mathrm{c}}} = 1 - p_{\mathrm{c}}(2).$$

We begin with the square lattice $\mathbb{L}^2$, for which $p_{\mathrm{c}}(2) = \sqrt{2}/(1+\sqrt{2})$. When $\beta = 0$, the model amounts to site percolation with density $\frac{1}{2}$. Since this percolation process has critical point satisfying $p_{\mathrm{c}}^{\mathrm{site}} > \frac{1}{2}$, each spin-cluster of the $\beta = 0$ Ising model is subcritical, and in particular has an exponentially-decaying tail. More specifically, write $x \overset{\pm}{\longleftrightarrow} y$ if there exists a path of $\mathbb{L}^2$ from x to y with constant spin-value, and let

$$S_x = \{y \in V : x \overset{\pm}{\longleftrightarrow} y\}$$

be the spin-cluster at x, and $S = S_0$. By the above, there exists $\alpha > 0$ such that

$$\lambda_0(|S| \ge n+1) \le e^{-\alpha n}, \qquad n \ge 1, \tag{8.37}$$

where λ_β denotes the infinite-volume Ising measure. It is standard (and follows from Theorem 8.17(a)) that there is a unique Gibbs measure for the Ising model when $\beta < \beta_{\mathrm{c}}$, and this may be extended to the critical case $\beta = \beta_{\mathrm{c}}$ (see [101] for example).

[8] rewrite

The exponential decay of (8.37) extends throughout the subcritical phase in the following sense. Yasunari Higuchi [123] has proved that

$$\lambda_\beta(|S| \ge n+1) \le e^{-\alpha n}, \qquad n \ge 1, \tag{8.38}$$

where $\alpha = \alpha(\beta)$ satisfies $\alpha > 0$ when $\beta < \beta_c$. There is a more recent proof of this (and more) by Rob van den Berg [31, Thm 2.4], using the sharp-threshold theorem, Theorem 4.77. Note that (8.38) implies the weaker (and known) statement that the clusters of the $q = 2$ random-cluster model on $\mathbb{L}^2$ have exponentially-decaying tail.[9]

Inequality (8.38) fails in an interesting manner when the square lattice is replaced by the triangular lattice $\mathbb{T}$. Since $p_c^{\text{site}}(\mathbb{T}) = \frac{1}{2}$, the $\beta = 0$ Ising model is critical. In particular, the tail of $|S|$ is of power-type and, by Smirnov's theorem for percolation, the scaling limit of the spin-cluster boundaries is SLE_6. Furthermore, the process is, in the following sense, *critical* for all $\beta \in [0, \beta_c]$. Since there is a unique Gibbs state for $\beta < \beta_c$, λ_β is invariant under the interchange of spin-values $-1 \leftrightarrow +1$. Let R_n be a rhombus of the lattice with side-lengths n and axes parallel to the horizontal and one of the diagonal lattice directions, and let A_n be the event that R_n is traversed from left to right by a $+$ path (i.e., a path ν satisfying $\sigma_y = +1$ for all $y \in \nu$). It is easily seen that the complement of A_n is the event that R_n is crossed from top to bottom by a $-$ path (see Figure 5.13 for an illustration of the analogous case of bond percolation on the square lattice). Therefore,

$$\lambda_\beta(A_n) = \tfrac{1}{2}, \qquad 0 \le \beta < \beta_c. \tag{8.39}$$

Let S_x be the spin-cluster containing x as before, and define

$$\text{rad}(S_x) = \max\{|z - x| : z \in S_x\},$$

where $|y|$ is the graph-theoretic distance from 0 to y. By (8.39), there exists a vertex x such that $\lambda_\beta(\text{rad}(S_x) \ge n) \ge (2n)^{-1}$. By the translation-invariance of λ_β,

$$\lambda_\beta(\text{rad}(S) \ge n) \ge \frac{1}{2n}, \qquad 0 \le \beta < \beta_c.$$

In conclusion, the tail of $\text{rad}(S)$ is of power-type for all $\beta \in [0, \beta_c)$.[10]

It is believed that the SLE_6 cluster-boundary limit 'propagates' from $\beta = 0$ all the way to $\beta < \beta_c$. When $\beta = \beta_c$, the corresponding limit is the same as that for the square lattice, namely SLE_3, see [62].

[9] Mention Graham-Grimmett

[10] Mention Balint-Camia-Meester

8.6 Random even graphs

We call a subset F of the edge-set of $G = (V, E)$ *even* if each vertex $x \in V$ is incident to an even number of elements of F, and we write $\mathcal{E}$ for the set of even subsets F. The subgraph (V, F) of G is *even* if F is even. It is standard that every even set F may be decomposed as an edge-disjoint union of cycles. Let $p \in [0, 1)$. The *random even subgraph* of G with parameter p is that with law

$$\eta_p(F) = \frac{1}{Z_{\mathrm{e}}} p^{|F|}(1-p)^{|E \setminus F|}, \qquad F \in \mathcal{E}, \tag{8.40}$$

where

$$Z_{\mathrm{e}} = \sum_{F \in \mathcal{E}} p^{|F|}(1-p)^{|E \setminus F|}.$$

When $p = \frac{1}{2}$, we talk of a *uniform* random even subgraph.

We may express η_p in the following way. Let $\phi_p = \phi_{p,1}$ be product measure with density p on $\Omega = \{0, 1\}^E$. For $\omega \in \Omega$, let $\partial\omega$ denote the set of vertices $x \in V$ that are incident to an odd number of ω-open edges. Then

$$\eta_p(F) = \frac{\phi_p(\omega_F)}{\phi_p(\partial\omega = \varnothing)}, \qquad F \in \mathcal{E},$$

where ω_F is the edge-configuration whose open set is F. In other words, ϕ_p describes the random subgraph of G obtained by randomly and independently deleting each edge with probability $1 - p$, and η_p is the law of this random subgraph conditioned on its being even.

Let λ_β be the Ising measure on a graph H with inverse temperature $\beta \geq 0$, presented in the form

$$\lambda_\beta(\sigma) = \frac{1}{Z_{\mathrm{I}}} \exp\left(\beta \sum_{e=\langle x,y \rangle \in E} \sigma_x \sigma_y\right), \qquad \sigma = (\sigma_x : x \in V) \in \Sigma, \tag{8.41}$$

with $\Sigma = \{-1, +1\}^V$. See (7.17) and (7.19). A spin configuration σ gives rise to a subgraph $G^\sigma = (V, E^\sigma)$ of G with E^σ given in (8.36) as the set of edges whose endpoints have like spin. When G is planar, the boundary of any connected component of G^σ corresponds to a cycle in the dual graph G_{d}, and the union of all such cycles is a (random) even subgraph of G_{d}. A glance at (8.3) informs us that the law of this even graph is η_r where

$$\frac{r}{1-r} = e^{-2\beta}.$$

Note that $r \leq \frac{1}{2}$. Thus, one way of generating a random even subgraph of a planar graph $G = (V, E)$ with parameter $r \in [0, \frac{1}{2}]$ is to take the dual of the graph G^σ with σ is chosen with law (8.41), and with $\beta = \beta(r)$ chosen suitably.

The above recipe may be cast in terms of the random-cluster model on the planar graph G. First, we sample ω according to the random-cluster measure $\phi_{p,q}$ with $p = 1-e^{-2\beta}$ and $q = 2$. To each open cluster of ω we allocate a random spin taken uniformly from $\{-1, +1\}$. These spins are constant on clusters and independent between clusters. By the discussion of Section 8.1, the resulting spin-configuration σ has law λ_β. The boundaries of the spin-clusters may be constructed as follows from ω. Let $C_1, C_2, \dots, C_c$ be the external boundaries of the open clusters of ω, and let $\xi_1, \xi_2, \dots, \xi_c$ be independent Bernoulli random variables with parameter $\frac{1}{2}$. The sum $\sum_i \xi_i C_i$, with addition interpreted as symmetric difference, has law η_r.

It turns out that one may generate a random even subgraph of a graph G from the random-cluster model on G, for an arbitrary, possibly non-planar, graph G. We consider first the uniform case of η_p with $p = \frac{1}{2}$.

We identify the family of all spanning subgraphs of $G = (V, E)$ with the family of all subsets of E (the word 'spanning' indicates that these subgraphs have the original vertex-set V). This family can further be identified with $\Omega = \{0, 1\}^E = \mathbb{Z}_2^E$, and is thus a vector space over $\mathbb{Z}_2$; the operation $+$ of addition is componentwise addition modulo 2, which translates into taking the symmetric difference of edge-sets: $F_1 + F_2 = F_1 \triangle F_2$ for $F_1, F_2 \subseteq E$.

The family $\mathcal{E}$ of even subgraphs of G forms a subspace of the vector space $\mathbb{Z}_2^E$, since $F_1 \triangle F_2$ is even if F_1 and F_2 are even. In particular, the number of even subgraphs of G equals $2^{c(G)}$ where $c(G) = \dim(\mathcal{E})$. The quantity $c(G)$ is thus the number of independent cycles in G, and, as is well known,

$$c(G) = |E| - |V| + k(G). \tag{8.42}$$

Cf. (8.26).

(8.43) Theorem [101]. *Let $C_1, C_2, \dots, C_c$ be a maximal set of independent cycles in G. Let $\xi_1, \xi_2, \dots, \xi_c$ be independent Bernoulli random variables with parameter $\frac{1}{2}$. Then $\sum_i \xi_i C_i$ is a uniform random even subgraph of G.*

Proof. Since every linear combination $\sum_i \psi_i C_i$, $\psi \in \{0, 1\}^c$, is even, and since every even graph may be expressed uniquely in this form, the uniform measure on $\{0, 1\}^c$ generates the uniform measure on $\mathcal{E}$. □

One standard way of choosing such a set $C_1, C_2, \dots, C_c$, when G is planar, is given as above by the external boundaries of the finite faces. Another is as follows. Let (V, F) be a *spanning subforest* of G, that is, the union of a spanning tree from each component of G. It is well known, and easy to check, that each edge $e_i \in E \setminus F$ can be completed by edges in F to a unique cycle C_i. These cycles form a basis of $\mathcal{E}$. By Theorem 8.43, we may therefore find a random uniform subset of the C_j by choosing a random uniform subset of $E \setminus F$.

We show next how to couple the $q = 2$ random-cluster model and the random even subgraph of G. Let $p \in [0, \frac{1}{2}]$, and let ω be a realization of the random-cluster model on G with parameters $2p$ and $q = 2$. Let $R = (V, \gamma)$ be a uniform random even subgraph of $(V, \eta(\omega))$.

(8.44) Theorem [101]. *The graph $R = (V, \gamma)$ is a random even subgraph of G with parameter p.*

This recipe for random even subgraphs provides a neat method for their simulation, provided $p \le \frac{1}{2}$. One may sample from the random-cluster measure by the method of coupling from the past (see [178]), and then sample a uniform random even subgraph from the outcome, as above. If G itself is even, we can further sample from η_p for $p > \frac{1}{2}$ by first sampling a subgraph $(V, \widetilde{F})$ from η_{1-p} and then taking the complement $(V, E \setminus \widetilde{F})$, which has the distribution η_p. One may adapt this argument to obtain an efficient method for sampling from η_p for $p > \frac{1}{2}$ and general G (see Exercise 8.16). When G is planar, this amounts to sampling from an antiferromagnetic Ising model on its dual graph.

There is a converse to Theorem 8.44. Take a random even subgraph (V, F) of $G = (V, E)$ with parameter $p \le \frac{1}{2}$. To each $e \notin F$, we assign an independent random colour, blue with probability $p/(1-p)$ and red otherwise. Let B be obtained from F by adding in all blue edges. It is left as an exercise to show that the graph (V, B) has law $\phi_{2p,2}$.

Proof of Theorem 8.44. Let $g \subseteq E$ be even, and let ω be a sample configuration of the random-cluster model on G. By the above,

$$\mathbb{P}(\gamma = g \mid \omega) = \begin{cases} 2^{-c(\omega)} & \text{if } g \subseteq \eta(\omega), \\ 0 & \text{otherwise,} \end{cases}$$

where $c(\omega) = c(V, \eta(\omega))$ is the number of independent cycles in the ω-open subgraph. Therefore,

$$\mathbb{P}(\gamma = g) = \sum_{\omega:\, g \subseteq \eta(\omega)} 2^{-c(\omega)} \phi_{2p,2}(\omega).$$

By (8.42),

$$\begin{aligned} \mathbb{P}(\gamma = g) &\propto \sum_{\omega:\, g \subseteq \eta(\omega)} (2p)^{|\eta(\omega)|}(1-2p)^{|E \setminus \eta(\omega)|} 2^{k(\omega)} \left(\tfrac{1}{2}\right)^{|\eta(\omega)|-|V|+k(\omega)} \\ &\propto \sum_{\omega:\, g \subseteq \eta(\omega)} p^{|\eta(\omega)|}(1-2p)^{|E \setminus \eta(\omega)|} \\ &= [p + (1-2p)]^{|E \setminus g|} p^{|g|} \\ &= p^{|g|}(1-p)^{|E \setminus g|}, \qquad g \subseteq E. \end{aligned}$$

The claim follows. □

The above account of even subgraphs would be gravely incomplete without a reminder of the so-called ‘random-current representation’ of the Ising model. This is a representation of the Ising measure in terms of a random field of loops and lines, and it has enabled a rigorous analysis of the Ising model. See [3, 7, 10] and [98, Chap. 9]. The random-current representation is closely related to the study of random even subgraphs.

8.7 Exercises

8.1. [107] Let $\phi_{p,q}$ be a random-cluster measure on a finite graph $G = (V, E)$ with parameters p and q. Prove that

$$\frac{d}{dp}\phi_{p,q}(A) = \frac{1}{p(1-p)}\Big\{\phi_{p,q}(M1_A) - \phi_{p,q}(M)\phi_{p,q}(A)\Big\}$$

for any event A, where $M = |\eta(\omega)|$ is the number of open edges of a configuration ω and 1_A is the indicator function of the event A.

8.2. (continuation) Show that $\phi_{p,q}$ satisfies the FKG inequality if $q \ge 1$, in that $\phi_{p,q}(A \cap B) \ge \phi_{p,q}(A)\phi_{p,q}(B)$ for increasing events A, B, but does not generally have this property when $q < 1$.

8.3. Show that the conditional random-cluster measure on G given that the edge e is closed (respectively, open) is that of $\phi_{G\setminus e,p,q}$ (respectively, $\phi_{G.e,p,q}$).

8.4. Show that random-cluster measures $\phi_{p,q}$ do not generally satisfy the BK inequality if $q > 1$. That is, find a finite graph G and increasing events A, B such that $\phi_{p,q}(A \circ B) > \phi_{p,q}(A)\phi_{p,q}(B)$.

8.5. (Important research problem, hard if true) Prove that random-cluster measures satisfy the BK inequality if $q < 1$.

8.6. Let $\phi_{p,q}$ be the random-cluster measure on a finite connected graph $G = (V, E)$. Show, in the limit as $p, q \to 0$ in such way that $q/p \to 0$, that $\phi_{p,q}$ converges weakly to the uniform spanning tree measure UST on G. Identify the corresponding limit as $p, q \to 0$ with $p = q$. Explain the relevance of these limits to the previous question.

8.7. [81] *Comparison inequalities.* Use the Holley inequality to prove the following 'comparison inequalities' for a random-cluster measure $\phi_{p,q}$ on a finite graph:

$$\phi_{p',q'} \le_{\rm st} \phi_{p,q} \qquad \text{if } q' \ge q,\ q' \ge 1,\ p' \le p,$$

$$\phi_{p',q'} \ge_{\rm st} \phi_{p,q} \qquad \text{if } q' \ge q,\ q' \ge 1,\ \frac{p'}{q'(1-p')} \ge \frac{p}{q(1-p)}.$$

8.8. [9] Show that the wired percolation probability $\theta^1(p, q)$ on $\mathbb{L}^d$ equals the limit of the finite-volume probabilities, in that, for $q \ge 1$,

$$\theta^1(p, q) = \lim_{\Lambda\to\mathbb{Z}^d} \phi^1_{\Lambda,p,q}(0 \leftrightarrow \partial\Lambda).$$

8.9. [98, 156] *Mixing.* A translation τ of $\mathbb{L}^d$ induces a translation of Ω given by $\tau(\omega)(e) = \omega(\tau^{-1}(e))$. Let A and B be cylinder events of Ω. Show, for $q \ge 1$ and $b = 0, 1$, that

$$\phi^b_{p,q}(A \cap \tau^n B) \to \phi^b_{p,q}(A)\phi^b_{p,q}(B) \qquad \text{as } n \to \infty.$$

The following may help when $b = 0$, with a similar argument when $b = 1$.

a. Assume A is increasing. Let A be defined on the box Λ, and let Δ be a larger box with $\tau^n B$ defined on $\Delta \setminus \Lambda$. Use positive association to show that

$$\phi^0_{\Delta,p,q}(A \cap \tau^n B) \geq \phi^0_{\Lambda,p,q}(A)\phi^0_{\Delta,p,q}(\tau^n B).$$

b. Let $\Delta \uparrow \mathbb{Z}^d$, and then $n \to \infty$ and $\Lambda \uparrow \mathbb{Z}^d$, to obtain

$$\liminf_{n\to\infty} \phi^0_{p,q}(A \cap \tau^n B) \geq \phi^0_{p,q}(A)\phi^0_{p,q}(B).$$

By applying this to the complement $\overline{B}$ also, deduce that $\phi^0_{p,q}(A \cap \tau^n B) \to \phi^0_{p,q}(A)\phi^b_{p,q}(B)$.

8.10. *Ergodicity.* Deduce from the result of the previous exercise that the $\phi^b_{p,q}$ are ergodic.

8.11. Use the comparison inequalities to prove that the critical point $p_c(q)$ of the random-cluster model on $\mathbb{Z}^d$ satisfies

$$p_c(1) \leq p_c(q) \leq \frac{q p_c(1)}{1 + (q-1)p_c(1)}, \qquad q \geq 1.$$

In particular, $0 < p_c(q) < 1$ if $q \geq 1$ and $d \geq 2$.

8.12. Let μ be the 'usual' coupling of the Potts measure and the random-cluster measure on a finite graph G. Derive the conditional measures of the first component given the second, and of the second given the first.

8.13. Let $q \in \{2, 3, \dots\}$, and let $G = (V, E)$ be a finite graph. Let $W \subseteq V$, and let $\sigma_1, \sigma_2 \in \{1, 2, \dots, q\}^W$. Starting from the random-cluster measure $\phi^W_{p,q}$ on G with members of W identified as a single point, explain how to couple the associated Potts measures $\pi(\cdot \mid \sigma_W = \sigma_i)$, for $i = 1, 2$, in such a way that: any vertex x not joined to W in the random-cluster configuration has the same spin in each of the two Potts configurations.

Let $B \subseteq \{1, 2, \dots, q\}^Y$ where $Y \subseteq V \setminus W$. Show that

$$\bigl|\pi(B \mid \sigma_W = \sigma_1) - \pi(B \mid \sigma_W = \sigma_2)\bigr| \leq \phi^W_{p,q}(W \leftrightarrow Y).$$

8.14. *Ising mixing and ergodicity.* Let $\phi^b_{p,q}$ be a random-cluster measure on $\mathbb{L}^d$ with $b \in \{0, 1\}$ and $q \in \{1, 2, \dots\}$. If $b = 0$, we assign a uniformly random element of $Q = \{1, 2, \dots, q\}$ to each open cluster, constant within clusters and independent between. We do similarly if $b = 1$ with the difference that any infinite cluster receives spin 1. Show that the ensuing spin-measures π^b are the infinite-volume Potts measures with free and 1 boundary conditions, respectively.

Using the results of the previous exercise, or otherwise, show that π^b is mixing, and hence ergodic, if $\phi^b_{p,q}(0 \leftrightarrow \infty) = 0$.

8.15. [93] Show for the random-cluster model on $\mathbb{L}^2$ that $p_c(q) \ge \kappa_q$, where $\kappa_q = \sqrt{q}/(1+\sqrt{q})$ is the self-dual point.

8.16. [101] Make a proposal for generating a random even subgraph of the graph $G = (V, E)$ with parameter p satisfying $p > \frac{1}{2}$.

You may find it useful to prove the following first. Let u, v be distinct vertices in the same component of G, and let π be a path from u to v. Let $\mathcal{F}$ be the set of even subsets of E, and $\mathcal{F}^{u,v}$ the set of subsets F such that $\deg_F(x)$ is even if and only if $x \ne u, v$. [Here, $\deg_F(x)$ is the number of elements of F incident to x.] Then $\mathcal{F}$ and $\mathcal{F}^{u,v}$ are put in one–one correspondence by $F \leftrightarrow F \triangle \pi$.

8.17. [101] Let (V, F) be a random even subgraph of $G = (V, E)$ with law η_p where $p \le \frac{1}{2}$. Each $e \notin F$ is coloured blue with probability $p/(1-p)$, independently of all other edges. Let B be the union of F with the blue edges. Show that (V, B) has law $\phi_{2p.2}$.

9

Quantum Ising Model

The quantum Ising model on a finite graph G may be transformed into a continuum random-cluster model on the set obtained by attaching a copy of the real line to each vertex of G. The ensuing representation of the Gibbs operator is susceptible to probabilistic analysis. One application is to an estimate of entanglement in the one-dimensional system.

9.1 The model

The quantum Ising model is defined as follows on the finite graph $G = (V, E)$. To each vertex $x \in V$ is associated a quantum spin-$\frac{1}{2}$ with local Hilbert space $\mathbb{C}^2$. The configuration space $\mathcal{H}$ for the system is the tensor product[1] $\mathcal{H} = \bigotimes_{x \in V} \mathbb{C}^2$. As basis for the copy of $\mathbb{C}^2$ labelled by $x \in V$, we take the two eigenvectors, denoted as

$$|+\rangle_x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |-\rangle_x = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

of the Pauli matrix

$$\sigma_x^{(3)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

at the site x, with corresponding eigenvalues ± 1. The other two Pauli matrices with respect to this basis are:

$$\sigma_x^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_x^{(2)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

In the following, $|\phi\rangle$ denotes a vector and $\langle\phi|$ its adjoint (or conjugate transpose).

Let D be the set of $2^{|V|}$ basis vectors $|\eta\rangle$ for $\mathcal{H}$ of the form $|\eta\rangle = \bigotimes_x |\pm\rangle_x$. There is a natural one–one correspondence between D and the space $\Sigma = \Sigma_V =$

[1] The tensor product $U \otimes V$ of two vector spaces over F is the dual space of the set of bilinear functionals on $U \times V$. Ref?

$\{-1, +1\}^V$. We may speak of members of Σ as basis vectors, and of $\mathcal{H}$ as the Hilbert space generated by Σ.

Let $\lambda, \delta \in [0, \infty)$. The Hamiltonian of the quantum Ising model with transverse field is the matrix (or 'operator')

$$H = -\tfrac{1}{2}\lambda \sum_{e=\langle x,y\rangle \in E} \sigma_x^{(3)}\sigma_y^{(3)} - \delta \sum_{x \in V} \sigma_x^{(1)}, \tag{9.1}$$

Here, λ is the spin-coupling and δ is the transverse-field intensity. The matrix H operates on vectors (elements of $\mathcal{H}$) through the operation of each σ_x on the component of the vector at x.

Let $\beta \in [0, \infty)$ be the parameter known as 'inverse temperature'. The Hamiltonian H generates the matrix $e^{-\beta H}$, and we are concerned with the operation of this matrix on elements of $\mathcal{H}$. We normalize $e^{-\beta H}$ by its trace, that is, we define the so-called 'density matrix'

$$\rho_G(\beta) = \frac{1}{Z_G(\beta)} e^{-\beta H}, \tag{9.2}$$

where

$$Z_G(\beta) = \mathrm{tr}(e^{-\beta H}) = \sum_{\eta \in \Sigma} \langle \eta | e^{-\beta H} | \eta \rangle. \tag{9.3}$$

It turns out that the matrix elements of $\rho_G(\beta)$ may be expressed in terms of a type of 'path integral' with respect to the continuum random-cluster model on $V \times [0, \beta]$ with parameters λ, δ and $q = 2$. We explain this in the following two sections.

The Hamiltonian H has a unique pure ground state $|\psi_G\rangle$ defined at zero-temperature (that is, in the limit as $\beta \to \infty$) as the eigenvector corresponding to the lowest eigenvalue of H.

9.2 Continuum percolation and random-cluster models

The finite graph $G = (V, E)$ may be used as a base for a family of probabilistic models that live not on the vertex-set V but on the 'continuum' space $V \times \mathbb{R}$. The simplest of these models is continuum percolation, see Section 6.6. We consider here a related model called the continuum random-cluster model. Let $\beta \in (0, \infty)$, and let Λ be the 'box' $\Lambda = V \times [0, \beta]$. In the notation of Section 6.6, let $\phi_{\Lambda,\lambda,\delta}$ denote the probability measure associated with the Poisson processes D_x, $x \in V$, and B_e, $e = \langle x, y\rangle \in E$. As sample space we take the set Ω_Λ comprising all finite sets of cuts and bridges in Λ, and we may assume without loss of generality that no cut is the endpoint of any bridge. For $\omega \in \Omega_\Lambda$, we write $B(\omega)$ and $D(\omega)$ for the sets of bridges and cuts, respectively, of ω. The appropriate σ-field $\mathcal{F}_\Lambda$ is that generated by the open sets in the associated Skorohod topology, see [34, 76].

For a given configuration $\omega \in \Omega_\Lambda$, let $k(\omega)$ be the number of its clusters under the connection relation $\leftrightarrow$. Let $q \in (0, \infty)$, and define the 'continuum random-cluster' probability measure $\phi_{\Lambda,\lambda,\delta,q}$ by

$$(9.4) \qquad d\phi_{\Lambda,\lambda,\delta,q}(\omega) = \frac{1}{Z} q^{k(\omega)} d\phi_{\Lambda,\lambda,\delta}(\omega), \qquad \omega \in \Omega_\Lambda,$$

for an appropriate normalizing constant, or 'partition function', $Z = Z_\Lambda(\lambda, \delta, q)$. The continuum random-cluster model may be studied in very much the same way as the random-cluster model on a lattice, see Chapter 8.

The space Ω_Λ is a partially ordered space with order relation given by: $\omega_1 \le \omega_2$ if $B(\omega_1) \subseteq B(\omega_2)$ and $D(\omega_1) \supseteq D(\omega_2)$. A random variable $X : \Omega_\Lambda \to \mathbb{R}$ is called *increasing* if $X(\omega) \le X(\omega')$ whenever $\omega \le \omega'$. An event $A \in \mathcal{F}_\Lambda$ is called *increasing* if its indicator function 1_A is increasing. Given two probability measures μ_1, μ_2 on a measurable pair $(\Omega_\Lambda, \mathcal{F}_\Lambda)$, we write $\mu_1 \le_{\mathrm{st}} \mu_2$ if $\mu_1(X) \le \mu_2(X)$ for all bounded increasing continuous random variables $X : \Omega_\Lambda \to \mathbb{R}$.

The measures $\phi_{\Lambda,\lambda,\delta,q}$ have certain properties of stochastic ordering as the parameters Λ, λ, δ, q vary. The basic theory will be assumed here, and the reader is referred to [37] for further details. In rough terms, the $\phi_{\Lambda,\lambda,\delta,q}$ inherit the properties of stochastic ordering and positive association enjoyed by their counterparts on discrete graphs. Of particular value in Section 9.5 is the stochastic inequality

$$(9.5) \qquad \phi_{\Lambda,\lambda,\delta,q} \le_{\mathrm{st}} \phi_{\Lambda,\lambda,\delta}, \qquad q \ge 1.$$

We note that the thermodynamic limit may be taken in much the same manner as it was for the discrete random-cluster model, whenever $q \ge 1$, and for certain boundary conditions τ. Suppose, for example, that V is a finite connected subgraph of the lattice $G = \mathbb{Z}^d$, and assign to the box $\Lambda = V \times [0, \beta]$ a suitable boundary condition. As described in [98] for the discrete case, if the boundary condition τ is chosen in such a way that the measures $\phi^\tau_{\Lambda,\lambda,\delta,q}$ are monotonic as $V \uparrow \mathbb{Z}^d$, then the weak limit $\phi^\tau_{\lambda,\delta,q,\beta} = \lim_{V\uparrow\mathbb{Z}^d} \phi^\tau_{\Lambda,\lambda,\delta,q}$ exists. One may similarly allow the limit as $\beta \to \infty$ to obtain a measure $\phi^\tau_{\lambda,\delta,q} = \lim_{\beta\to\infty} \phi^\tau_{\lambda,\delta,q,\beta}$.

Let $G = \mathbb{Z}^d$. Restricting ourselves for convenience to the case of free boundary conditions, we define the percolation probability by

$$\theta(\lambda, \delta, q) = \phi_{\lambda,\delta,q}(|C| = \infty),$$

where C is the cluster at the origin $(0, 0)$, and $|C|$ denotes the aggregate (one-dimensional) Lebesgue measure of the time intervals comprising C. The critical point is defined by

$$\lambda_c(\mathbb{Z}^d, q) = \sup\{\lambda : \theta(\lambda, 1, q) = 0\}.$$

In the special case $d = 1$, the random-cluster model has a property of self-duality that leads to the following conjecture.

(9.6) Conjecture. *The continuum random-cluster model on $\mathbb{Z} \times \mathbb{R}$ with cluster-weighting factor satisfying $q \ge 1$ has critical value $\lambda_c(\mathbb{Z}, q) = q$.*

It may be proved by standard means that $\lambda_c(\mathbb{Z}, q) \ge q$. See (8.32) and [98, Sect. 6.2] for the corresponding result on the discrete lattice $\mathbb{Z}^2$. The cases $q = 1, 2$ are special. The statement $\lambda_c(\mathbb{Z}, 1) = 1$ is part of Theorem 6.19(b). When $q = 2$, the method of so-called 'random currents' may be adapted to the quantum model with several consequences, of which we highlight the fact that $\lambda_c(\mathbb{Z}, 2) = 2$; see [38].

The *continuum Potts model* on $V \times \mathbb{R}$ is given as follows. Let $q \in \{2, 3, \dots\}$. To each cluster of the random-cluster model with cluster-weighting factor q is assigned a 'spin' from the space $\Sigma = \{1, 2, \dots, q\}$, different clusters receiving independent spins. The outcome is a function $\sigma : V \times \mathbb{R} \to \Sigma$, and this is the spin-vector of a 'continuum q-state Potts model' with parameters λ and δ. When $q = 2$, we refer to the model as a *continuum Ising model*.

It is not hard to see that the law $\mathbb{P}$ of the above spin model on $\Lambda = V \times [0, \beta]$ is given by

$$d\mathbb{P}(\sigma) = \frac{1}{Z} e^{\lambda L(\sigma)}\, d\phi_{\Lambda,\delta}(D_\sigma),$$

where D_σ is the set of $(x, s) \in V \times [0, \beta]$ such that $\sigma(x, s-) \ne \sigma(x, s+)$, $\phi_{\Lambda,\delta}$ is the law of a family of independent Poisson processes on the time-lines $\{x\} \times [0, \beta]$, $x \in V$, with intensity δ, and

$$L(\sigma) = \sum_{\langle x,y \rangle \in E_V} \int_0^\beta 1_{\{\sigma(x,u)=\sigma(y,u)\}}\, du$$

is the aggregate Lebesgue measure of those subsets of pairs of adjacent time-lines on which the spins are equal. As usual, Z is an appropriate constant.

9.3 Quantum Ising via random-cluster

In this section we describe the relationship between the quantum Ising model on a finite graph $G = (V, E)$ and the continuum random-cluster model on $G \times [0, \beta]$ with $q = 2$. We shall see that the density matrix $\rho_G(\beta)$ may be expressed in terms of ratios of probabilities given in terms of the random-cluster model. The roots of the following argument lie in the work of Campanino, von Dreyfus, Klein, and Perez, and the reader is referred to [13] for the final form. Similar geometrical transformations can be made for other certain quantum models, see [14, 167].

First, some notation. Let $\Lambda = V \times [0, \beta]$, and let Ω_Λ be the configuration space of the continuum random-cluster model on Λ. For given λ, δ, and $q = 2$, let $\phi_{G,\beta}$ denote the corresponding continuum random-cluster measure on Ω_Λ (with free boundary conditions). Thus, for economy of notation we suppress reference to λ and δ.

We next introduce a coupling of edge and spin configurations as in Section 8.1. For $\omega \in \Omega_\Lambda$, let $S(\omega)$ denote the space of all functions $s : V \times [0, \beta] \to \{-1, +1\}$ that are constant on the clusters of ω, and let S be the union of the $S(\omega)$ over $\omega \in \Omega_\Lambda$. Given ω, we may pick an element of $S(\omega)$ uniformly at random, and we denote this random element as σ. We shall abuse notation by using $\phi_{G,\beta}$ to denote the ensuing probability measure on the coupled space $\Omega_\Lambda \times S$. For $s \in S$ and $W \subseteq V$, we write $s_{W,0}$ (respectively, $s_{W,\beta}$) for the vector $(s(x, 0) : x \in W)$ (respectively, $(s(x, \beta) : x \in W)$). We abbreviate $s_{V,0}$ and $s_{V,\beta}$ to s_0 and s_β, respectively.

(9.7) Theorem [13]. *The elements of the density matrix $\rho_G(\beta)$ satisfy*

$$\langle \eta' | \rho_G(\beta) | \eta \rangle = \frac{\phi_{G,\beta}(\sigma_0 = \eta,\ \sigma_\beta = \eta')}{\phi_{G,\beta}(\sigma_0 = \sigma_\beta)}, \qquad \eta, \eta' \in \Sigma. \tag{9.8}$$

Proof. We use the notation of Section 9.1. By (9.1) with $\nu = \frac{1}{2}\sum_{\langle x,y\rangle} \lambda \mathbb{I}$ and $\mathbb{I}$ the identity matrix[2],

$$e^{-\beta(H+\nu)} = e^{-\beta(U+V)}, \tag{9.9}$$

where

$$U = -\delta \sum_{x\in V} \sigma_x^{(1)}, \qquad V = -\tfrac{1}{2} \sum_{e=\langle x,y\rangle \in E} \lambda(\sigma_x^{(3)}\sigma_y^{(3)} - \mathbb{I}).$$

Although these two matrices do not commute, we may use the so-called Lie–Trotter formula (see, for example, [193]) to express $e^{-\beta(U+V)}$ in terms of single-site and two-site contributions due to U and V, respectively. By the Lie–Trotter formula,

$$e^{-(U+V)\Delta t} = e^{-U\Delta t} e^{-V\Delta t} + \mathrm{O}(\Delta t^2).$$

We divide the interval $[0, \beta]$ into N parts each of length $\Delta t = 1/N$, so that

$$e^{-\beta(U+V)} = \lim_{\Delta t \to 0} (e^{-U\Delta t} e^{-V\Delta t})^{\beta/\Delta t}.$$

Now expand the exponential, neglecting terms of order $\mathrm{o}(\Delta t)$, to obtain

(9.10)

$$e^{-\beta(H+\nu)} = \lim_{\Delta t\to 0}\left(\prod_x \left[(1-\delta\Delta t)\mathbb{I} + \delta\Delta t P_x^1\right] \prod_{e=\langle x,y\rangle} \left[(1-\lambda\Delta t)\mathbb{I} + \lambda\Delta t P_{x,y}^3\right]\right)^{\beta/\Delta t},$$

where $P_x^1 = \sigma_{(x)}^1 + \mathbb{I}$ and $P_{x,y}^3 = \frac{1}{2}(\sigma_x^{(3)}\sigma_y^{(3)} + \mathbb{I})$.

[2]Note that $\langle \eta' | e^{J + c\mathbb{I}} | \eta \rangle = e^c \langle \eta' | e^J | \eta \rangle$, so the introduction of ν into the exponent is harmless.

As noted earlier, we think of members of $\Sigma = \{-1, +1\}^V$ as basis vectors of $\mathcal{H}$, and of $\mathcal{H}$ as the Hilbert space generated by Σ. The product (9.10) contains a collection of operators acting on sites x and on neighbouring pairs $\langle x, y\rangle$. We partition the time interval $[0, \beta]$ into N time-segments labelled $\Delta t_1, \Delta t_2, \dots, \Delta t_N$, each of length Δt. On neglecting terms of order $o(\Delta t)$, we may see that each given time-segment arising in (9.10) contains exactly one of: the identity matrix $\mathbb{I}$; a matrix of the form P_x^1; a matrix of the form $P_{x,y}^3$. Each such matrix occurs in the time-segment with a certain weight.

Let us consider the actions of these matrices on the states $|\eta\rangle$ for each time interval Δt_i, $i \in \{1, 2, \dots, N\}$. The matrix elements of the single-site operator at x are given by

$$\langle \eta' | \sigma_x^{(1)} + \mathbb{I} | \eta \rangle \equiv 1. \tag{9.11}$$

This is easily checked by exhaustion. When this matrix occurs in some time-segment Δt_i, we place a mark in the interval $\{x\} \times \Delta t_i$, and we call this mark a *cut*. Such a cut has a corresponding weight $\delta \Delta t + o(\Delta t)$.

The matrix element involving the neighbouring pair $\langle x, y\rangle$ yields, as above,

$$\tfrac{1}{2}\langle \eta' | \sigma_x^{(3)} \sigma_y^{(3)} + \mathbb{I} | \eta \rangle = \begin{cases} 1 & \text{if } \eta_x = \eta_y = \eta'_x = \eta'_y, \\ 0 & \text{otherwise.} \end{cases} \tag{9.12}$$

When this occurs in some time-segment Δt_i, we place a *bridge* between the intervals $\{x\} \times \Delta t_i$ and $\{y\} \times \Delta t_i$. Such a bridge has a corresponding weight $\lambda \Delta t + o(\Delta t)$.

In the limit $\Delta t \to 0$, the spin operators generate thus a Poisson process with intensity δ of cuts in each time-line $\{x\} \times [0, \beta]$, and a Poisson process with intensity λ of bridges between each pair $\{x\} \times [0, \beta]$, $\{y\} \times [0, \beta]$ of time-lines, for neighbouring x and y. This is an independent family of Poisson processes. We write D_x for the set of cuts at the site x, and B_e for the set of bridges corresponding to an edge $e = \langle x, y\rangle$. The configuration space is the set Ω_Λ containing all finite sets of cuts and bridges, and we may assume without loss of generality that no cut is the endpoint of any bridge.

For two points $(x, s), (y, t) \in \Lambda$, we write $(x, s) \leftrightarrow (y, t)$ if there exists a cut-free path from the first to the second that traverses time-lines and bridges. A *cluster* is a maximal subset C of Λ such that $(x, s) \leftrightarrow (y, t)$ for all $(x, s), (y, t) \in C$. Thus the connection relation $\leftrightarrow$ generates a continuum percolation process on Λ, and we write $\phi_{\Lambda,\lambda,\delta}$ for the probability measure corresponding to the weight function on the configuration space Ω_Λ. That is, $\phi_{\Lambda,\lambda,\delta}$ is the measure governing a family of independent Poisson processes of cuts (with intensity δ) and of bridges (with intensity λ). The ensuing percolation process has appeared in Section 6.6.

Equations (9.11)–(9.12) are to be interpreted in the following way. In calculating the operator $e^{-\beta(H+\nu)}$, one averages over contributions from realizations of the Poisson processes, on the basis that the quantum spins are constant on every

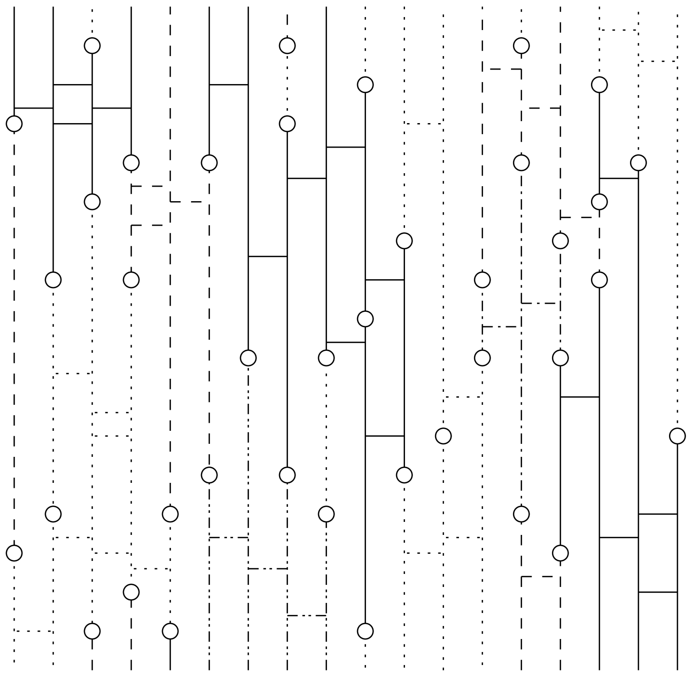

Figure 9.1. An example of a space–time configuration contributing to the Poisson integral (9.18). The cuts are shown as circles and the distinct connected clusters are indicated with different line-types.

cluster of the corresponding percolation process, and each such spin-function is equiprobable.

More explicitly,

$$e^{-\beta(H+\nu)} = \int d\phi_{\Lambda,\lambda,\delta}(\omega)\left(\mathcal{T}\prod_{(x,t)\in D} P_x^1(t) \prod_{(\langle x,y\rangle,t')\in B} P_{x,y}^3(t')\right), \tag{9.13}$$

where $\mathcal{T}$ denotes the time-ordering of the terms in the products, and B (respectively, D) is the set of all bridges (respectively, cuts) of the configuration $\omega \in \Omega_\Lambda$.

Let $\omega \in \Omega_\Lambda$. Let μ_ω be the counting measure on the space $S(\omega)$ of functions $s : V \times [0, \beta] \to \{-1, +1\}$ that are constant on the clusters of ω. Let $K(\omega)$ be the time-ordered product of operators in (9.13). We may evaluate the matrix elements of $K(\omega)$ by inserting the 'resolution of the identity'

$$\sum_{\eta\in\Sigma} |\eta\rangle\langle\eta| = \mathbb{I} \tag{9.14}$$

between any two factors in the product, obtaining thus that

$$\langle\eta'|K(\omega)|\eta\rangle = \sum_{s\in S(\omega)} 1_{\{s_0=\eta\}} 1_{\{s_\beta=\eta'\}}, \qquad \eta, \eta' \in \Sigma. \tag{9.15}$$

This is the number of spin-allocations to the clusters of ω with given spin-vectors at times 0 and β.

The matrix elements of $\rho_G(\beta)$ are therefore given by

(9.16) $$\langle \eta' | \rho_G(\beta) | \eta \rangle = \frac{1}{Z_{G,\beta}} \int 1_{\{s_0=\eta\}} 1_{\{s_\beta=\eta'\}} \, d\mu_\omega(s) \, d\phi_{\Lambda,\lambda,\delta}(\omega),$$

for $\eta, \eta' \in \Sigma$, where

(9.17) $$Z_{G,\beta} = \mathrm{tr}(e^{-\beta(H+\nu)}).$$

For $\eta, \eta' \in \Sigma$, let $I_{\eta,\eta'}$ be the indicator function of the event (in Ω_Λ) that, for all $x, y \in V$,

$$\text{if } (x,0) \leftrightarrow (y,0), \text{ then } \eta_x = \eta_y,$$
$$\text{if } (x,\beta) \leftrightarrow (y,\beta), \text{ then } \eta'_x = \eta'_y,$$
$$\text{if } (x,0) \leftrightarrow (y,\beta), \text{ then } \eta_x = \eta'_y.$$

This is the event that the pair (η, η') of initial and final spin-vectors is 'compatible' with the random-cluster configuration. We have that

(9.18) $$\begin{aligned} \langle \eta' | \rho_G(\beta) | \eta \rangle &= \frac{1}{Z_{G,\beta}} \int d\phi_{\Lambda,\lambda,\delta}(\omega) \sum_{s \in \Sigma(\omega)} 1_{\{s_0=\eta\}} 1_{\{s_\beta=\eta'\}} \\ &= \frac{1}{Z_{G,\beta}} \int 2^{\overline{k}(\omega)+\underline{k}(\omega)} I_{\eta,\eta'} \left(\tfrac{1}{2}\right)^{\underline{k}(\omega)} d\phi_{\Lambda,\lambda,\delta}(\omega) \\ &= \frac{1}{Z_{G,\beta}} \phi_{G,\beta}(\sigma_0 = \eta, \, \sigma_\beta = \eta'). \qquad \eta, \eta' \in \Sigma, \end{aligned}$$

where $\overline{k}(\omega)$ is the number of clusters of ω containing no point of the form $(v, 0)$ or (v, β), for $v \in V$, and $\underline{k}(\omega) = k(\omega) - \overline{k}(\omega)$ is the number remaining. See Figure 9.1 for an illustration of the space–time configurations contributing to the Poisson integral (9.18).

On setting $\eta = \eta'$ in (9.18) and summing over $\eta \in \Sigma$, we find that

(9.19) $$Z_{G,\beta} = \phi_{G,\beta}(\sigma_0 = \sigma_\beta),$$

as required. □

This section closes with an alternative expression for the formula of Theorem 9.7. We consider 'periodic' boundary conditions on Λ obtained by, for each $x \in V$, identifying $(x, 0)$ and (x, β). Let $k^{\mathrm{per}}(\omega)$ be the number of open clusters of ω with periodic boundary conditions, and $\phi^{\mathrm{per}}_{G,\beta}$ be the corresponding random-cluster measure. By (9.18),

$$\langle \eta' | \rho_G(\beta) | \eta \rangle = \frac{1}{Z_{G,\beta}} \int 2^{\overline{k}(\omega)} I_{\eta,\eta'} \, d\phi_{\Lambda,\lambda,\delta}(\omega).$$

By setting $\eta' = \eta$ and summing,

$$(9.20) \qquad 1 = \sum_{\eta \in \Sigma} \langle \eta | \rho_G(\beta) | \eta \rangle = \frac{1}{Z_{G,\beta}} \int 2^{\overline{k}(\omega)} 2^{k^{\text{per}}(\omega) - \overline{k}(\omega)} \, d\phi_{\Lambda,\lambda,\delta}(\omega),$$

whence $Z_{G,\beta}$ equals the normalizing constant for the periodic random-cluster measure $\phi^{\text{per}}_{G,\beta}$.

9.4 Long-range order

The two-point connectivity function

$$\tau_{G,\beta}(x,y) = \phi^{\text{per}}_{G,\beta}\big((x,0) \leftrightarrow (y,0)\big), \qquad x, y \in V,$$

for the periodic random-cluster measure turns out to be a natural measure of long-range order in the quantum Ising model, with the order parameter of the latter given as in the next theorem.

(9.21) Theorem [13]. *We have that*

$$\tau_{G,\beta}(x,y) = \operatorname{tr}\big(\rho_G(\beta)\sigma^{(3)}_x \sigma^{(3)}_y\big), \qquad x, y \in V.$$

Proof. The argument leading to (9.18) is easily adapted to obtain

$$\operatorname{tr}\big(\rho_G(\beta) \cdot \tfrac{1}{2}(\sigma^{(3)}_x \sigma^{(3)}_y + \mathbb{I})\big) = \frac{1}{Z_{G,\beta}} \int 2^{\overline{k}(\omega)} \Bigg(\sum_{\eta:\, \eta_x = \eta_y} I_{\eta,\eta} \Bigg) \, d\phi_{\Lambda,\lambda,\delta}(\omega).$$

Now,

$$\sum_{\eta:\, \eta_x = \eta_y} I_{\eta,\eta} = \begin{cases} 2^{k^{\text{per}}(\omega) - \overline{k}(\omega)} & \text{if } (x,0) \leftrightarrow (y,0), \\ 2^{k^{\text{per}}(\omega) - \overline{k}(\omega) - 1} & \text{if } (x,0) \not\leftrightarrow (y,0), \end{cases}$$

whence, by the remark at the end of the last section,

$$\operatorname{tr}\big(\rho_G(\beta) \cdot \tfrac{1}{2}(\sigma^{(3)}_x \sigma^{(3)}_y + \mathbb{I})\big) = \tau_{G,\beta}(x,y) + \tfrac{1}{2}(1 - \tau_{G,\beta}(x,y)),$$

and the claim follows. □

Expand discussion of critical point?

The infinite-volume limits of the quantum Ising model on G are obtained in the 'ground state' as $\beta \to \infty$, and in the spatial limit as $|V| \to \infty$. The paraphernalia of the discrete random-cluster model may be adapted to the current continuous setting in order to understand the issues of existence and uniqueness of these limits.

We do not investigate that here. Instead, we point out that the behaviour of the two-point connectivity function, after taking the limits $\beta \to \infty$, $|V| \to \infty$, depends pivotally on the existence or not of an unbounded cluster in the random-cluster model. Let $\phi_{\lambda,\delta}$ be the infinite-volume measure, and let

$$\theta(\lambda, \delta) = \phi_{\lambda,\delta}(C_0 \text{ is unbounded})$$

be the percolation probability. Then $\tau_{\lambda,\delta}(x, y) \to 0$ as $|x - y| \to \infty$, when $\theta(\lambda, \delta) = 0$. On the other hand, by the FKG inequality and the (a.s.) uniqueness of the unbounded cluster,

$$\tau_{\lambda,\delta}(x, y) \geq \theta(\lambda, \delta)^2,$$

implying that $\tau_{\lambda,\delta}(x, y)$ is bounded uniformly away from 0 when $\theta(\lambda, \delta) > 0$.

A more detailed investigation of the infinite-volume limits and their implications for the quantum Ising model may be found in [13]. As pointed out there, the situation is more interesting in the 'disordered' setting, when the λ_e and δ_x are themselves random variables.

9.5 Entanglement in one dimension

It is shown next how the random-cluster analysis of the last section enables progress with the problem of so-called entanglement in one dimension. The principle reference for the work of this section is [106].

Let $G = (V, E)$ be a finite graph, and let $W \subseteq V$. A considerable effort has been spent on understanding the so-called 'entanglement' of the spins in W relative to those of $V \setminus W$, in the (ground state) limit as $\beta \to \infty$. This is already a hard problem when G is a finite subgraph of the line $\mathbb{Z}$. Various methods have been used in this case, and a variety of results, some rigorous, obtained.

The first step in the definition of entanglement is to define the *reduced density matrix*

$$\rho_G^W(\beta) = \operatorname{tr}_{V \setminus W}(\rho_G(\beta)),$$

where the trace is taken over the Hilbert space $\mathcal{H}_{V \setminus W} = \bigotimes_{x \in V \setminus W} \mathbb{C}^2$ of spins of vertices of $V \setminus W$. An analysis (omitted here) exactly parallel to that leading to Theorem 9.7 allows the following representation of the matrix elements of $\rho_G^W(\beta)$.

(9.22) Theorem [106]. *The elements of the reduced density matrix $\rho_G^W(\beta)$ satisfy*

$$(9.23) \qquad \langle \eta' | \rho_G^W(\beta) | \eta \rangle = \frac{\phi_{G,\beta}(\sigma_{W,0} = \eta,\ \sigma_{W,\beta} = \eta' \mid E)}{\phi_{G,\beta}(\sigma_0 = \sigma_\beta \mid E)}, \qquad \eta, \eta' \in \Sigma_W,$$

where E is the event that $\sigma_{V \setminus W,0} = \sigma_{V \setminus W,\beta}$.

Let D_W be the set of $2^{|W|}$ vectors $|\eta\rangle$ of the form $|\eta\rangle = \bigotimes_{x \in W} |\pm\rangle_x$, and write $\mathcal{H}_W$ for the Hilbert space generated by D_W. Just as before, there is a natural one–one correspondence between D_W and the space $\Sigma_W = \{-1, +1\}^W$, and we shall thus regard $\mathcal{H}_W$ as the Hilbert space generated by Σ_W.

We may write

$$\rho_G = \lim_{\beta\to\infty} \rho_G(\beta) = |\psi_G\rangle\langle\psi_G|$$

for the density matrix corresponding to the ground state of the system, and similarly

(9.24) $$\rho_G^W = \mathrm{tr}_{V\setminus W}(|\psi_G\rangle\langle\psi_G|) = \lim_{\beta\to\infty} \rho_G^W(\beta).$$

The entanglement of the spins in W may be defined as follows.

(9.25) Definition. The *entanglement* of the vertex-set W relative to its complement $V \setminus W$ is the entropy

(9.26) $$S_G^W = -\mathrm{tr}(\rho_G^W \log_2 \rho_G^W).$$

The behaviour of S_G^W, for general G and W, is not understood at present. We specialize here to the case of a finite subset of the one-dimensional lattice $\mathbb{Z}$. Let $m, L \ge 0$ and take $V = [-m, m+L]$ and $W = [0, L]$, viewed as subsets of $\mathbb{Z}$. We obtain G from V by adding edges between each pair $x, y \in V$ with $|x - y| = 1$. We write $\rho_m(\beta)$ for $\rho_G(\beta)$, and S_m^L (respectively, ρ_m^L) for S_G^W (respectively, ρ_G^W). A key step in the study of S_m^L for large m is a bound on the norm of the difference $\rho_m^L - \rho_n^L$. The *operator norm* of a Hermitian matrix[3] A is given by

$$\|A\| = \sup_{\|\psi\|=1} \bigl|\langle\psi|A|\psi\rangle\bigr|,$$

where the supremum is over all vectors ψ with L^2-norm 1.

(9.27) Theorem [106]. *Let $\lambda, \delta \in (0, \infty)$ and write $\theta = \lambda/\delta$. There exist constants C, α, γ depending on θ and satisfying $\gamma > 0$ when $\theta < 1$ such that*:

(9.28) $$\|\rho_m^L - \rho_n^L\| \le \min\bigl\{2, CL^\alpha e^{-\gamma m}\bigr\}, \qquad 2 \le m \le n < \infty,\ L \ge 1.$$

One would expect that γ may be taken in such a manner that $\gamma > 0$ under the weaker assumption $\lambda/\delta < 2$, but this has not yet quite been proved (cf. Conjecture 9.6). The constant γ is, apart from a constant factor, the reciprocal of the correlation length of the associated random-cluster model.

Proved, no?

Inequality (9.28) is proved by the following route. Consider the continuum random-cluster model with $q = 2$ on the space–time graph $\Lambda = V \times [0, \beta]$ with 'partial periodic top/bottom boundary conditions'; that is, for each $x \in V \setminus W$,

[3] A matrix is called *Hermitian* if it equals its conjugate transpose.

we identify the two vertices $(x, 0)$ and (x, β). Let $\phi^{\mathrm{p}}_{m,\beta}$ denote the associated random-cluster measure on Ω_Λ. To each cluster of $\omega \in \Omega_\Lambda$ we assign a random spin from $\{-1, +1\}$ in the usual manner, and we abuse notation by using $\phi^{\mathrm{p}}_{m,\beta}$ to denote the measure governing both the random-cluster configuration and the spin configuration. Let

$$a_{m,\beta} = \phi^{\mathrm{p}}_{m,\beta}(\sigma_{W,0} = \sigma_{W,\beta}),$$

noting that

$$a_{m,\beta} = \phi_{m,\beta}(\sigma_0 = \sigma_\beta \mid E)$$

as in (9.23).

By Theorem 9.22,

(9.29)

$$\langle \psi | \rho^L_m(\beta) - \rho^L_n(\beta) | \psi \rangle = \frac{\phi^{\mathrm{p}}_{m,\beta}(c(\sigma_{W,0})\overline{c(\sigma_{W,\beta})})}{a_{m,\beta}} - \frac{\phi^{\mathrm{p}}_{n,\beta}(c(\sigma_{W,0})\overline{c(\sigma_{W,\beta})})}{a_{n,\beta}},$$

where $c : \{-1, +1\}^W \to \mathbb{C}$ and

$$\psi = \sum_{\eta \in \Sigma_W} c(\eta)\eta \in \mathcal{H}_W.$$

The random-cluster property of ratio weak-mixing is used in the derivation of (9.28) from (9.29). This may be stated roughly as follows. Let A and B be events in the continuum random-cluster model that are defined on regions R_A and R_B of space, respectively. What can be said about the difference $\phi(A \cap B) - \phi(A)\phi(B)$ when the distance $d(R_A, R_B)$ between R_A and R_B is large? It is not hard to show that this difference is exponentially small in the distance, so long as the random-cluster model has exponentially-decaying connectivities, and such a property is called 'weak mixing'. It is harder to show a similar bound for the difference $\phi(A \mid B) - \phi(A)$, and such a bound is termed 'ratio weak-mixing'. The ratio weak-mixing property of random-cluster model has been investigated in [18, 19] for the discrete case and in [106] for the continuum model.

At the final step of the proof of Theorem 9.27, the random-cluster model is compared via (9.5) with the continuum percolation model of Section 6.6, and the exponential decay of Theorem 9.27 follows by Theorem 6.19. A logarithmic bound on the entanglement entropy follows for sufficiently small λ/δ.

(9.30) Theorem [106]. *Let $\lambda, \delta \in (0, \infty)$ and write $\theta = \lambda/\delta$. There exists $\theta_0 \in (0, \infty)$ such that: for $\theta < \theta_0$, there exists $K = K(\theta) < \infty$ such that*

$$S^L_m \le K \log_2 L, \qquad m \ge 0,\ L \ge 2.$$

Discuss: spectra are close, so...

A stronger result is expected, namely that the entanglement S^L_m is bounded above, uniformly in L, whenever θ is sufficiently small, and perhaps for all $\theta < \theta_c$

where $\theta_c = 2$ is the critical point. It is not clear whether this is provable by the methods of this chapter. See Conjecture 9.6 above, and the references in [106].

There is no rigorous picture known of the behaviour of S_m^L for large θ, or of the corresponding quantity in dimensions $d \geq 2$, although Theorem 9.27 has a counterpart in this setting. Theorem 9.30 may be extended to the disordered system in which the intensities λ, δ are independent random variables indexed by the vertices and edges of the underlying graph, subject to certain conditions on these variables (cf. Theorem 6.20 and the preceding discussion).

9.6 Exercises

9.1. Explain in what manner the continuum random-cluster measure $\phi_{\lambda,1,q}$ on $\mathbb{Z} \times \mathbb{R}$ is 'self-dual' when $\lambda = q$ and $q \geq 1$.

10

Interacting Particle Systems

The contact, voter, and exclusion models are examples of so-called interacting particle systems. Each is a Markov process in continuous time, with state space $\{0, 1\}^V$ for some countable set V. In the voter model, each element of V may be in either of two states, and its state flips at a rate that is a weighted average of the states of the other elements. When $V = \mathbb{Z}^d$, the analysis of the voter model hinges on the recurrence or transience of an associated Markov chain. When $d = 2$ and the model is generated by simple random walk, the only invariant measures are the two point masses on the (two) states representing unanimity. The picture is more complicated when $d \geq 2$, owing to the transience of the random walk. In the exclusion model, a set of particles moves about V according to a 'symmetric' Markov chain, subject to exclusion. We shall assume that $V = \mathbb{Z}^d$, and that the Markov chain is translation-invariant. It turns out that the product measures are invariant for this process, and furthermore that these are exactly the extremal invariant measures.

10.1 Introductory remarks

There are many beautiful problems of physical type that may be modelled as Markov processes on the compact state space $\Sigma = \{0, 1\}^V$ for some countable set V. Amongst the most studied to date by probabilists are the contact, voter, and exclusion models[1]. This significant branch of modern probability theory had its nascence around 1970 in the work of Roland Dobrushin, Frank Spitzer, and others, and has been brought to maturity through the work of Thomas Liggett and colleagues. The basic references are Liggett's two volumes [148, 150], see also [151].

The general theory of Markov processes, with its intrinsic complexities, is avoided here. The three processes of this chapter may be constructed via 'graphical representations' involving independent random walks. There is a general approach

[1] We say nothing about the stochastic Ising model here.

to such important matters as existence, for an account of which the reader is referred to [148]. The two observations of note are that the state space Σ is compact, and that the Markov processes $(\eta_t : t \geq 0)$ of this section are Feller processes, which is to say that the transition measures are weakly continuous functions of the initial state[2].

For a given Markov process, the two main questions are to identify the set of invariant measures, and to identify the 'basin of attraction' of a given invariant measure. The processes of this chapter will always possess a non-empty set $\mathcal{I}$ of invariant measures, although it is not always possible to describe all members of this set explicitly. Since $\mathcal{I}$ is a convex set of measures, it suffices to describe its extremal elements. We shall see that, in certain circumstances, $|\mathcal{I}| = 1$, and this may be interpreted as the absence of long-range order.

Since V is infinite, Σ is uncountable. We normally specify the transition operators of a Markov chain on such Σ by specifying its *generator*. This is an operator G acting on an appropriate dense subset of $C(\Sigma)$, the space of continuous functions on Σ endowed with the product topology and the supremum norm. It is determined by its values on the space $\mathcal{C}(\Sigma)$ of cylinder functions, being the set of functions that depend on only finitely many coordinates in Σ. For $f \in \mathcal{C}(\Sigma)$, we write Gf in the form

$$(10.1) \qquad Gf(\eta) = \sum_{\eta' \in \Sigma} c(\eta, \eta')[f(\eta') - f(\eta)], \qquad \eta \in \Sigma,$$

for some function c. For $\eta \neq \eta'$, we think of $c(\eta, \eta')$ as being the rate at which the process, when in state η, jumps to state η'.

The processes η_t possesses a transition semigroup $(S_t : t \geq 0)$ acting on $C(\Sigma)$ and given by

$$(10.2) \qquad S_t f(\eta) = E^\eta(f(\eta_t)), \qquad \eta \in \Sigma,$$

where E^η denotes expectation under the assumption $\eta_0 = \eta$. Under certain conditions on the process, the transition semigroup is related to the generator by the formula

$$(10.3) \qquad S_t = \exp(tG),$$

suitably interpreted according to the Hille–Yosida theorem, see [148, Sect. I.2]. The semigroup acts on probability measures by

$$(10.4) \qquad \mu S_t(A) = \int_\Sigma P^\eta(\eta_t \in A)\, d\mu(\eta).$$

[2]Let $C(\Sigma)$ denote the space of continuous functions on Σ endowed with the product topology and the supremum norm. The process η_t is Feller if: for $f \in C(\Sigma)$, $f_t(\eta) = E^\eta(f(\eta_t))$ defines a function belonging to $C(\Sigma)$. Here, E^η denotes expectation with initial state η.

A probability measure μ on Σ is called *invariant* for the process η_t if $\mu S_t = \mu$ for all t. Under suitable conditions, μ is invariant if and only if

$$\int Gf \, d\mu = 0 \qquad \text{for all } f \in \mathcal{C}(\Sigma). \tag{10.5}$$

In the remainder of this chapter we shall encounter certain constructions of Markov processes on Σ, and all such constructions will satisfy the conditions alluded to above.

10.2 Contact model

Let $G = (V, E)$ be a connected graph with bounded vertex-degrees. The state space is $\Sigma = \{0, 1\}^V$, where the local state 1 (respectively, 0) represents 'ill' (respectively, 'healthy'). Ill vertices recover at rate δ, and healthy vertices become ill at a rate that is linear in the number of ill neighbours. See Chapter 6.

One proceeds more formally as follows. For $\eta \in \Sigma$ and $x \in V$, let η_x denote the state obtained from η by flipping the local state of x. That is,

$$\eta_x(y) = \begin{cases} 1 - \eta(x) & \text{if } y = x, \\ \eta(y) & \text{otherwise.} \end{cases}$$

We let the function c of (10.1) be given by

$$c(\eta, \eta_x) = \begin{cases} \delta & \text{if } \eta(x) = 1, \\ \lambda|\{y \sim x : \eta(y) = 1\}| & \text{if } \eta(x) = 0, \end{cases}$$

where λ and δ are strictly positive constants. If $\eta' = \eta_x$ for no $x \in V$, and $\eta' \neq \eta$, we set $c(\eta, \eta') = 0$.

We saw in Chapter 6 that the point mass on the empty set, $\underline{\nu} = \delta_\varnothing$, is the minimal invariant measure of the process, and that there exists a maximal invariant measure $\overline{\nu}$ obtained as the weak limit of the process with initial state V. As remarked at the end of Section 6.3, when $G = \mathbb{L}^d$, the set of extremal invariant measures is exactly $\mathcal{I}_\mathrm{e} = \{\delta_\varnothing, \overline{\nu}\}$, and $\delta_\varnothing = \overline{\nu}$ if and only if there is no percolation in the associated oriented percolation model in continuous time. Of especial use in proving these facts was the coupling of contact models in terms of Poisson processes of cuts and (directed) bridges.

We revisit duality briefly, see Theorem 6.1. For $\eta \in \Sigma$ and $A \subseteq V$, let

$$H(\eta, A) = \prod_{x \in A} [1 - \eta(x)] = \begin{cases} 1 & \text{if } \eta(x) = 0 \text{ for all } x \in A, \\ 0 & \text{otherwise.} \end{cases} \tag{10.6}$$

The conclusion of Theorem 6.1 may be expressed more generally as:

$$E^A(H(A_t, B)) = E^B(H(A, B_t)),$$

where A_t (respectively, B_t) denotes the contact model with initial state $A_0 = A$ (respectively, $B_0 = B$). This may seem a strange way to express the duality relation, but its relevance may become clearer soon.

10.3 Voter model

Let V be a countable set, and let $P = (p_{x,y} : x, y \in V)$ be the transition matrix of a Markov chain on V. The associated voter model is given by choosing

$$(10.7) \qquad c(\eta, \eta_x) = \sum_{y:\,\eta(y)\neq\eta(x)} p_{x,y}$$

in (10.1). The meaning of this is as follows. Each member of V is an individual in a population, and may have either of two opinions at any given time. Let $x \in V$. At times of a rate-1 Poisson process, x selects a random y according to the measure $p_{x,y}$, and adopts the opinion of y. It turns out that the behaviour of this model is closely related to the transience/recurrence of the chain with transition matrix matrix P, and of properties of its harmonic functions.

The voter model has two absorbing states, namely all 0 and all 1, and we denote by δ_0 and δ_1 the point masses on these states. Any convex combination of δ_0 and δ_1 is invariant also, and thus one asks for conditions under which every invariant measure is of this form. A duality relation will enable us to answer this question.

It is helpful to draw the graphical representation of the process. With each $x \in V$ is associated a 'time-line' $[0, \infty)$, and on each such time-line is marked the set of epochs of a Poisson process Po_x with intensity 1. Different time-lines possess independent Poisson processes. Associated with each epoch of the Poisson process at x is a vertex y chosen at random according to the transition matrix P. The meaning of y is as above.

Consider the state of vertex x at time t. We imagine a particle that is at position x at time t, and we write $X_x(0) = x$. When we follow the time-line $x \times [0, t]$ backwards in time, that is, from (x, t) to $(x, 0)$, we encounter a first point (first in this reversed ordering of time) in Po_x. At this time the particle jumps to the selected neighbour of x. Continuing likewise, the particle performs a simple random walk about V. Writing $X_x(t)$ for its position at time 0, the (voter) state of x at time t is precisely that of $X_x(t)$ at time 0.

Suppose we proceed likewise starting from two vertices x and y at time t. Tracing the states of x and y backwards, each follows a Markov chain with transition matrix P, denoted X_x and X_y respectively. These chains are independent until the first time (if ever) at which they meet. When they meet, they 'coalesce': if they ever occupy the same vertex at any given time, then they follow the same trajectory subsequently.

We state this as follows. The presentation here is somewhat informal, and may be made more complete as in [148]. We write $(\eta_t : t \geq 0)$ for the voter process, and $\mathcal{S}$ for the set of finite subsets of V.

(10.8) Theorem. *Let $A \in \mathcal{S}$, $\eta \in \Sigma$, and let $(A_t : t \geq 0)$ be a system of coalescing random walks beginning on the set $A_0 = A$. Then,*

$$\mathbb{P}^\eta(\eta_t \equiv 1 \text{ on } A) = \mathbb{P}^A(\eta \equiv 1 \text{ on } A_t), \qquad t \geq 0.$$

This may be expressed in the form

$$\mathbb{E}^{\eta}(H(\eta_t, A)) = \mathbb{E}^{A}(H(\eta, A_t)),$$

with

$$H(\eta, A) = \prod_{x \in A} \eta(x).$$

Proof. Each side of the equation is the measure of the complement of the event that, in the graphical representation, there is a path from $(x, 0)$ to (a, t) for some x with $\eta(x) = 0$ and some $a \in A$. □

For simplicity, we restrict ourselves henceforth to a case of special interest, namely with V the vertex-set $\mathbb{Z}^d$ of the d-dimensional lattice with $d \geq 1$, and with $p_{x,y} = p(x - y)$ for some function p. In the special case of simple random walk, where

$$p(z) = \frac{1}{2d}, \qquad |z| = 1, \tag{10.9}$$

we have that $\eta(x)$ flips at a rate equal to the proportion of neighbours of x whose states disagree with the current value $\eta(x)$. The case of general P is treated in [148].

Let X_t and Y_t be independent random walks on $\mathbb{Z}^d$ with rate-1 exponential holding times, and jump distribution $p_{x,y} = p(y - x)$. The difference $X_t - Y_t$ is a Markov chain also. If $X_t - Y_t$ is recurrent, we say that we are in the *recurrent case*, otherwise the *transient case*. The analysis of the voter model is fairly simple in the recurrent case.

(10.10) Theorem. *Assume we are in the recurrent case.*

1. $\mathcal{I}_{\mathrm{e}} = \{\delta_0, \delta_1\}$.
2. *If μ is a probability measure on Σ with $\mu(\eta(x) = 1) = \alpha$ for all $x \in \mathbb{Z}^d$, then $\mu S_t \Rightarrow (1 - \alpha)\delta_0 + \alpha\delta_1$ as $t \to \infty$.*

The situation is quite different in the transient case. We may construct a family of distinct invariant measures ν_α indexed by $\alpha \in [0, 1]$, and we do this as follows. Let ϕ_α be product measure on Σ with density α. We shall show the existence of the weak limits $\nu_\alpha = \lim_{t\to\infty} \phi_\alpha S_t$, and it turns out that the ν_α are exactly the extremal invariant measures. A partial proof of the next theorem is provided below.

(10.11) Theorem. *Assume we are in the transient case.*

1. *The weak limits $\nu_\alpha = \lim_{t\to\infty} \phi_\alpha S_t$ exist.*
2. *The ν_α are translation-invariant and ergodic*[3]*, with density*

$$\nu_\alpha(\eta(x) = 1) = \alpha.$$

[3] A probability measure μ on Σ is *ergodic* if any shift-invariant event has μ-probability either 0 or 1. It is standard that the ergodic measures are extremal within the class of translation-invariant measures, see [86] for example.

3. $\mathcal{I}_e = \{\nu_\alpha : \alpha \in [0,1]\}$.

We return briefly to the voter model corresponding to simple random walk on $\mathbb{Z}^d$, see (10.9). It is an elementary consequence of Pólya's theorem, Theorem 1.32, that we are in the recurrent case if and only $d \le 2$.

Proof of Theorem 10.10. By assumption, we are in the recurrent case. Let $x, y \in \mathbb{Z}^d$. By duality, Theorem 10.8, and recurrence,

$$\mathbb{P}(\eta_t(x) \ne \eta_t(y)) \le \mathbb{P}\big(X_x(u) \ne X_y(u) \text{ for } 0 \le u \le t\big) \to 0 \qquad \text{as } t \to \infty. \tag{10.12}$$

For $A \in \mathcal{S}$, $A \ne \varnothing$,

$$\mathbb{P}(\eta_t \text{ is non-constant on } A) \le \mathbb{P}^A(|A_t| > 1),$$

and, by (10.12),

$$\mathbb{P}^A(|A_t| > 1) \le \sum_{x,y\in A} \mathbb{P}\big(X_x(u) \ne X_y(u) \text{ for } 0 \le u \le t\big) \to 0 \qquad \text{as } t \to \infty. \tag{10.13}$$

It follows that, for any extremal invariant measure μ, the μ-measure of the set of constant configurations is 1. Only the convex combinations of δ_0 and δ_1 have this property.

Let μ be a probability measure with density α, as in the statement of the theorem, and let $A \in \mathcal{S}$, $A \ne \varnothing$. By Theorem 10.8 again,

$$\begin{aligned}
\mu S_t(\{\eta : \eta \equiv 1 \text{ on } A\}) &= \int \mathbb{P}^\eta(\eta_t \equiv 1 \text{ on } A)\, \mu(d\eta) \\
&= \int \mathbb{P}^A(\eta \equiv 1 \text{ on } A_t)\, \mu(d\eta) \\
&= \int \mathbb{P}^A(\eta \equiv 1 \text{ on } A_t,\ |A_t| > 1)\, \mu(d\eta) \\
&\qquad + \sum_{y\in\mathbb{Z}^d} \mathbb{P}^A(A_t = \{y\})\mu(\eta(y) = 1),
\end{aligned}$$

whence

$$\Big|\mu S_t(\{\eta : \eta \equiv 1 \text{ on } A\}) - \alpha\Big| \le 2\mathbb{P}^A(|A_t| > 1).$$

By (10.13), $\mu S_t \Rightarrow (1-\alpha)\delta_0 + \alpha\delta_1$ as claimed. □

Partial proof of Theorem 10.11. For $A \in \mathcal{S}$, $A \ne \varnothing$,

$$\begin{aligned}
\phi_\alpha S_t(\eta_t \equiv 1 \text{ on } A) &= \int \mathbb{P}^\eta(\eta_t \equiv 1 \text{ on } A)\, \phi_\alpha(d\eta) \\
&= \int \mathbb{P}^A(\eta \equiv 1 \text{ on } A_t)\, \phi_\alpha(d\eta) \\
&= \mathbb{E}^A(\alpha^{|A_t|}).
\end{aligned} \tag{10.14}$$

The last expectation converges as $t \to \infty$, since $|A_t|$ is non-increasing in t. Using the inclusion–exclusion principle, we deduce that the μS_t-measure of any cylinder event has a limit, and therefore the weak limit ν_α exists (see the discussion of weak convergence in Section 2.3). Since the initial state ϕ_α is translation-invariant, so is ν_α. We omit the proof of ergodicity, which may be found in [148, 151]. By (10.14) with $A = \{x\}$, $\phi_\alpha S_t(\eta(x) = 1) = \alpha$ for all t.

It may be shown that $\mathcal{I}$ is exactly the convex hull of the set $\{\nu_\alpha : \alpha \in [0, 1]\}$. Since the ν_α are ergodic, they are extremal within the class of translation-invariant measures, and therefore

$$\mathcal{I}_e \supseteq \{\nu_\alpha : \alpha \in [0, 1]\}. \tag{10.15}$$

Conversely, take $\nu \in \mathcal{I}_e$. By the remark above, ν is a mixture of the ν_α. Since ν is extremal, it equals one of the ν_α, whence equality holds in (10.15). The proof of the main statement above is omitted, and may be found in [148, 151]. □

10.4 Exclusion model

In this model for a lattice gas, particles jump around the countable set V, subject to the excluded-volume constraint that no more than one particle may occupy any given vertex at any given time. The state space is $\Sigma = \{0, 1\}^V$, where the local state 1 represents occupancy by a particle. The dynamics are assumed to proceed as follows. Let $P = (p_{x,y} : x, y \in V)$ be the transition matrix of a Markov chain on V. In order to guarantee the existence of the corresponding exclusion process, we shall assume that

$$\sup_{y \in V} \sum_{x \in V} p_{x,y} < \infty.$$

If the current state is $\eta \in \Sigma$, and $\eta(x) = 1$, the particle at x waits an exponentially distributed time, parameter 1, before it attempts to jump. At the end of this holding time, it chooses a vertex y according to the probabilities $p_{x,y}$. If, at this instant, y is empty then this particle jumps to y. If y is occupied, the jump is suppressed, and the particle remains at x. Particles are deemed to be indistinguishable.

The generator G of the Markov process is given by

$$Gf(\eta) = \sum_{\substack{x,y \in V: \\ \eta(x)=1,\, \eta(y)=0}} p_{x,y}[f(\eta_{x,y}) - f(\eta)],$$

for cylinder functions f, where $\eta_{x,y}$ is the state obtained from η by interchanging the local states of x and y, that is,

$$\eta_{x,y}(z) = \begin{cases} \eta(x) & \text{if } z = y, \\ \eta(y) & \text{if } z = x, \\ \eta(z) & \text{otherwise.} \end{cases}$$

We may construct the process via a graphical representation, as in Section 10.3. For each $x \in V$, we let Po_x be a Poisson process with rate 1; these are the times at which a particle at x (if, indeed, x is occupied at the relevant time) attempts to move away from x. With each 'time' $T \in \text{Po}_x$, we associate a vertex Y chosen according to the mass function $p_{x,y}$, $y \in V$. If x is occupied by a particle at time T, this particle attempts to jump at this instant of time to the new position Y. The jump is successful if Y is empty at time T, otherwise the move is suppressed.

It is immediate that the two Dirac measures δ_0 and δ_1 are invariant. We shall see below that the family of invariant measures is generally much richer than this. The theory is substantially simpler in the *symmetric* case, and thus we assume henceforth that

$$p_{x,y} = p_{y,x}, \qquad x, y \in V. \tag{10.16}$$

See [148, Chap. VIII] and [151] for bibliographies for the asymmetric case. If V is the vertex-set of a graph $G = (V, E)$, and P is the transition matrix of simple random walk on G, then (10.16) amounts to the assumption that G be regular.

Mention TASEP etc.

Duality plays once again a central role in the analysis of the symmetric process. We shall see that the model is self-dual, in the sense of the following Theorem 10.17. Note first that the graphical representation of a symmetric model may be expressed in a slightly simplified manner. For each unordered pair $x, y \in V$, let $\text{Po}_{x,y}$ be a Poisson process with intensity $p_{x,y}$ $[= p_{y,x}]$. For each $T \in \text{Po}_{x,y}$, we interchange the states of x and y at time T. That is, any particle at x moves to y, and vice versa. It is easily seen that the corresponding particle system is the exclusion model. For every $x \in V$, a particle at x at time 0 would pursue a trajectory through V that is determined by the graphical representation, and we denote this trajectory by $R_x(t)$, $t \ge 0$, noting that $R_x(0) = x$. The processes $R_x(\cdot)$, $x \in V$, are of course dependent.

The family $(R_x(\cdot) : x \in V)$ is time-reversible in the following sense. Let $t > 0$ be given. For each $y \in V$, one may trace the trajectory arriving at (y, t) backwards in time, and we denote the resulting path by $B_{y,t}(v)$, $0 \le v \le t$, with $B_{y,t}(0) = y$. It is clear by the properties of a Poisson process that the families $(R_x(u) : u \in [0, t],\ x \in V)$ and $(B_{y,t}(v) : v \in [0, t],\ y \in V)$ have the same laws.

Let $(\eta_t : t \ge 0)$ denote the exclusion model. We distinguish the general model from one that possesses only finitely many particles. Let $\mathcal{S}$ be the set of finite subsets of V, and write $(A_t : t \ge 0)$ for an exclusion process with initial state $A_0 \in \mathcal{S}$. We think of η_t as a random 0/1-vector, and of A_t as a random subset of the vertex-set V.

(10.17) Theorem. *Consider a symmetric exclusion model on V. For every $\eta \in \Sigma$ and $A \in \mathcal{S}$,*

$$\mathbb{P}^\eta(\eta_t \equiv 1 \text{ on } A) = \mathbb{P}^A(\eta \equiv 1 \text{ on } A_t), \qquad t \ge 0. \tag{10.18}$$

Proof. The left side of (10.18) equals the probability that, in the graphical representation: for every $y \in A$ there exists $x \in V$ with $\eta(x) = 1$ such that $R_x(t) = y$. By the remarks above, this equals the probability that $\eta(R_y(t)) = 1$ for every $y \in A$. □

(10.19) Corollary. *Consider a symmetric exclusion model on V. For each $\alpha \in [0, 1]$, the product measure ϕ_α on Σ is invariant.*

Refs here and elsewhere.

Proof. Let η be sampled from Σ according to the product measure ϕ_α. We have that

$$\mathbb{P}^A(\eta \equiv 1 \text{ on } A_t) = \alpha^{|A_t|} = \alpha^{|A|}.$$

By Theorem 10.17, if η_0 has law ϕ_α, then so does η_t for all t. That is, ϕ_α is an invariant measure. □

The question thus arises of determining the circumstances under which the set of invariant extremal measures is *exactly* the set of product measures.

Assume for simplicity that: $V = \mathbb{Z}^d$, that the transition probabilities are symmetric and translation-invariant in that

$$p_{x,y} = p_{y,x} = p(y - x), \qquad x, y \in \mathbb{Z}^d,$$

for some function p, and that the associated Markov chain is irreducible. It can be shown in this case (see [148, 151]) that $\mathcal{I}_e = \{\phi_\alpha : \alpha \in [0, 1]\}$, and that

$$\mu S_t \Rightarrow \phi_\alpha \qquad \text{as } t \to \infty,$$

for any translation-invariant and spatially ergodic probability measure μ with $\alpha = \mu(\eta(0) = 1)$.

In the more general symmetric *non-translation-invariant* case on an arbitrary countable set V, the constants α are replaced by the set $\mathcal{H}$ of functions $\alpha : V \to [0, 1]$ satisfying

$$\alpha(x) = \sum_{y \in V} p_{x,y}\alpha(y), \qquad x \in V, \tag{10.20}$$

that is, the bounded harmonic functions, re-scaled if necessary to take values in $[0, 1]$. [Recall that an irreducible symmetric translation-invariant Markov chain on $\mathbb{Z}^d$ has only *constant* bounded harmonic functions[4].] Let μ_α be the product measure on Σ with $\mu_\alpha(\eta(x) = 1) = \alpha(x)$. It turns out that the weak limit

$$\nu_\alpha = \lim_{t \to \infty} \mu_\alpha S_t$$

[4] *Exercise*: Prove this statement. It is an easy consequence of the optional stopping theorem for bounded martingales, whenever the chain is recurrent. See [148, pp. 67–70] for a discussion of the general case.

exists, and that $\mathcal{I}_e = \{\nu_\alpha : \alpha \in \mathcal{H}\}$. It may be shown that: ν_α is a product measure if and only if α is a constant function. See [148, 151].

One may find examples for which the set $\mathcal{H}$ is large. Let $P = (p_{x,y})$ be the transition matrix of simple random walk on a binary tree T (each of whose vertices has degree 3, see Figure 6.2). Let 0 be a given vertex of the tree, and think of 0 as the root of three disjoint sub-trees of T. Any solution $(a_n : n \geq 0)$ to the difference equation

$$2a_{n+1} - 3a_n + a_{n-1} = 0, \qquad n \geq 1, \tag{10.21}$$

defines a harmonic function α on a given such sub-tree, by $\alpha(x) = a_n$ where n is the distance between 0 and x. The general solution to (10.21) is

$$a_n = A + B(\tfrac{1}{2})^n,$$

where A and B are arbitrary constants. The three pairs (A, B), corresponding to the three sub-trees at 0, may be chosen in an arbitrary manner, subject to the condition that $a_0 = A + B$ is constant between sub-trees. Furthermore, the composite harmonic function on T takes values in $[0, 1]$ if and only if each pair (A, B) satisfies $A, A + B \in [0, 1]$. There is thus a continuum of admissible non-constant solutions to (10.20), and therefore a continuum of corresponding extremal invariant measures of the associated exclusion model.

10.5 Exercises

10.1. [212] *Biased voter model.* Each point of the square lattice is occupied, at each time t, by either a benign or a malignant cell. Benign cells invade their neighbours, each neighbour being invaded at rate β, and similarly malignant cells invade their neighbours at rate μ. Suppose there is exactly one malignant cell at time 0, and let $\kappa = \mu/\beta \geq 1$. Show that the malignant cells die out with probability κ^{-1}.

What happens on $\mathbb{Z}^d$ with $d \geq 2$?

10.2. *Stochastic Ising model.* Let $\Sigma = \{-1, +1\}^V$ be the state space of a Markov process on the finite graph $G = (V, E)$ in which the state at $x \in V$ changes value at rate $c(x, \sigma)$ when the state overall is σ. Show that the flip rates

$$c(x, \sigma) = \exp\Bigl(-\sum_{y \in \partial x} \sigma_x \sigma_y\Bigr),$$

$$c'(x, \sigma) = \frac{1}{1 + \exp\bigl(\sum_{y \in \partial x} 2\sigma_x \sigma_y\bigr)},$$

give rise to time-reversible dynamics with respect to the Ising measure with the same value of β and zero external-field.

10.3. A probability measure μ on $\{0, 1\}^{\mathbb{Z}}$ is called *exchangeable* if the quantity $\mu(\{\eta : \eta \equiv 1 \text{ on } A\})$, as A ranges over the set of finite subsets of $\mathbb{Z}$, depends only on the cardinality of A. Show that every exchangeable μ is invariant for a symmetric exclusion model on $\mathbb{Z}$.

11

Random Graphs

In the Erdős–Rényi random graph $G_{n,p}$, each pair of vertices is connected by an edge with probability p. We describe the emergence of the giant component when $pn \approx 1$, and identify the density of this component as the survival probability of a Poisson branching process. The Hoeffding inequality may be used to show that, for constant p, the chromatic number of $G_{n,p}$ is asymptotic to $\frac{1}{2}n/\log_\pi n$ where $\pi = 1/(1-p)$.

11.1 Erdős–Rényi graphs

Let $V = \{1, 2, \dots, n\}$, and let $(X_{i,j} : 1 \le i < j \le n)$ be independent Bernoulli random variables with parameter p. For each pair $i < j$ we place an undirected edge $\langle i, j\rangle$ between vertices i and j if and only if $X_{i,j} = 1$. The resulting random graph is named after Erdős and Rényi [75][1], and it is commonly denoted $G_{n,p}$. The density p of edges may vary with n, for example, $p = \lambda/n$ where $\lambda \in (0, \infty)$, and one commonly considers the structure of $G_{n,p}$ in the limit as $n \to \infty$.

The original motivation for studying $G_{n,p}$ was to understand the properties of 'typical' graphs. This is in contrast to the study of 'extremal' graphs, although it may be noted that random graphs have on occasion manifested properties more extreme than graphs obtained by more constructive means.

Random graphs have proved an important tool in the study of the 'typical' runtime of algorithms. Consider a computational problem associated with graphs, such as the travelling salesman problem. In assessing the speed of an algorithm for this problem, one may find that, in the worst situation, the algorithm is very slow. On the other hand, the typical runtime may be much less than the worst-case runtime. The measurement of 'typical' runtime requires a probability measure on the space of graphs, and it is in this regard that $G_{n,p}$ has risen to prominence within this subfield of theoretical computer science. While $G_{n,p}$ is, in a sense, the obvious candidate for such a probability measure, it suffers from the weakness

[1] See also [88].

that the 'mother graph' K_n has a large automorphism group; it is a poor candidate in situations in which pairs of vertices may have differing relationships to one another.

The random graph $G_{n,p}$ has received a very great deal of attention, largely within the community working on probabilistic combinatorics. The theory is based on a mix of combinatorial and probabilistic techniques, and has become very refined.

One may think of $G_{n,p}$ as a percolation model on the complete graph K_n. The parallel with percolation is weak in the sense that the theory of $G_{n,p}$ is largely combinatorial rather than geometrical. There is however a sense in which random graph theory has enriched percolation. The major difficulty in the study of physical systems arises out of the geometry of $\mathbb{R}^d$; points are related to one another in ways that depend greatly on their relative positions in $\mathbb{R}^d$. In a so-called 'mean-field theory', the geometrical component is removed through the assumption that points interact with all other points equally. Mean-field theory leads to a approximate picture of the model in question, and this approximation improves in the limit as $d \to \infty$. The Erdős–Rényi random graph may be seen as a mean-field approximation to percolation. Mean-field models based on $G_{n,p}$ have proved of value for Ising and Potts models also, see [41, 215].

The two principal references for the theory of $G_{n,p}$ are the earlier book [40] by Bollobás, and the more recent work [129] of Janson, Łuzcak and Ruciński. We say nothing here about recent developments in random-graph theory involving models for the so-called small world. See [72] for example.

11.2 Giant component

Consider the random graph $G_{n,\lambda/n}$ where $\lambda \in (0, \infty)$ is a constant. We may build the component containing a given vertex v as follows. The vertex v is adjacent to a certain number N of vertices, where N has the $\text{bin}(n-1, \lambda/n)$ distribution. Each of these vertices is joined to a random number of vertices, distributed approximately as N, and such that, with probability $1 - o(1)$, these new vertex-sets are disjoint. Since the $\text{bin}(n-1, \lambda/n)$ distribution is 'nearly' Poisson $\text{Po}(\lambda)$, the component at v grows very much like a branching process with family-size distribution $\text{Po}(\lambda)$. The branching-process approximation becomes less good as the component grows, and in particular when its size becomes of order n. The mean family-size equals λ, and thus the process with $\lambda < 1$ is very different from that with $\lambda > 1$.

Suppose that $\lambda < 1$. In this case, the branching process is (almost surely) extinct, and possesses a finite number of vertices. Having built the component at v, one picks another vertex w and acts similarly. By iteration, one obtains that $G_{n,p}$ is the union of clusters each with exponentially decaying tail. The largest component has order $\log n$.

When $\lambda > 1$, the branching process grows beyond limits with strictly positive probability. This corresponds to the existence in $G_{n,p}$ of a component of size having order n. We make this more formal as follows. Let X_n be the number of vertices in a largest component of $G_{n,p}$. We write $Z_n = \mathrm{o_p}(y_n)$ if $Z_n/y_n \to 0$ in probability as $n \to \infty$. An event A_n is said to occur *asymptotically almost surely* (abbreviated as a.a.s.) if $P(A_n) \to 1$ as $n \to \infty$.

(11.1) Theorem [75]. *We have that*

$$\frac{1}{n}X_n = \begin{cases} \mathrm{o_p}(1) & \textit{if } \lambda \le 1, \\ \alpha(\lambda)(1+\mathrm{o_p}(1)) & \textit{if } \lambda > 1, \end{cases}$$

where $\alpha(\lambda)$ *is the survival probability of a branching process with a single progenitor and family-size distribution* $\mathrm{Po}(\lambda)$.

It is standard (see [109, Sect. 5.4], for example) that the extinction probability $\eta(\lambda) = 1 - \alpha(\lambda)$ of such a branching process is the smallest non-negative root of the equation $s = G(s)$ where $G(s) = e^{\lambda(s-1)}$. It is left as an exercise[2] to check that

$$\eta(\lambda) = \frac{1}{\lambda}\sum_{k=1}^{\infty}\frac{k^{k-1}}{k!}(\lambda e^{-\lambda})^k.$$

Proof. Since the distribution of X_n is non-decreasing in λ, and since $\alpha(1) = 0$, it suffices to consider the case $\lambda > 1$, and we assume this henceforth. We follow [129, Sect. 5.2], and use a branching-process argument. (See also [20].) Choose a vertex v. At the first step we find all neighbours of v, say $v_1, v_2, \dots, v_r$, and we mark v as *dead*. At the second step, we generate all neighbours of v_1 in $V \setminus \{v, v_1, v_2, \dots, v_r\}$, and we mark v_1 as dead. This process is iterated until the entire component of $G_{n,p}$ containing v has been generated. Any vertex thus discovered in the component of v, but not yet dead, is said to be *live*. Step i is said to be complete when there are exactly i dead vertices.

Conditional on the history of the process up to and including the $(i-1)$th step, the number N_i of vertices added at step i is distributed as $\mathrm{bin}(n-m, p)$ where m is the number of vertices already generated.

Let

$$k_- = \frac{16\lambda}{(\lambda-1)^2}\log n, \qquad k_+ = n^{2/3}.$$

In this section, all logarithms are natural. Consider the above process started at v, and let A_v be the event that: either the process terminates after fewer than k_-

[2] Here is one way that resonates with random graphs. Let p_k be the probability that vertex 1 lies in a component that is a tree of size k. By enumerating the possibilities,

$$p_k = \binom{n-1}{k-1}k^{k-2}\left(\frac{\lambda}{n}\right)^{k-1}\left(1-\frac{\lambda}{n}\right)^{k(n-k)+\binom{k}{2}-k+1}.$$

Simplify and sum over k.

steps, or, for every k satisfying $k_- \le k \le k_+$, there are at least $\frac{1}{2}(\lambda-1)k$ live vertices after step k. If A_v does not occur, there exists $k \in [k_-, k_+]$ such that: step k takes place and, after its completion, fewer than

$$m = k + \tfrac{1}{2}(\lambda-1)k = \tfrac{1}{2}(\lambda+1)k$$

vertices have been discovered in all. Thus, on $\overline{A}_v$, and with such a choice for k,

$$(N_1, N_2, \ldots, N_k) \ge_{\text{st}} (Y_1, Y_2, \ldots, Y_k)$$

where the Y_j are independent random variables distributed as bin$(n - \frac{1}{2}(\lambda+1)k,\ p)$. Therefore,

$$1 - P(A_v) \le \sum_{k=k_-}^{k_+} \pi_k,$$

where, by the Chernoff bound for the tail of the binomial distribution, for $k_- \le k \le k_+$ and large n,

$$\begin{aligned} \pi_k &= P\left(\sum_{i=1}^{k} Y_i \le \tfrac{1}{2}(\lambda+1)k\right) \\ &\le \exp\left(-\frac{(\lambda-1)^2k^2}{9\lambda k}\right) \le \exp\left(-\frac{(\lambda-1)^2}{9\lambda}k_-\right) \\ &= \mathrm{O}(n^{-16/9}). \end{aligned} \tag{11.2}$$

Therefore, $1 - P(A_v) \le k_+ \mathrm{O}(n^{-16/9}) = \mathrm{o}(n^{-1})$, and this proves that

$$P\left(\bigcap_{v\in V} A_v\right) \ge 1 - \sum_{v\in V}[1 - P(A_v)] \to 1 \qquad \text{as } n \to \infty.$$

In particular, a.a.s., no component of $G_{n,\lambda/n}$ has size between k_- and k_+.

We show next that, a.a.s., there do not exist more than two components with size exceeding k_+. Assume that $\bigcap_v A_v$ occurs, and let v', v'' be distinct vertices lying in components with size exceeding k_+. We run the above process beginning at v' for the first k_+ steps, and we finish with a set L' containing at least $\frac{1}{2}(\lambda-1)k_+$ live vertices. We do the same for the process from v''. Either the growing component at v'' intersects the current component v' by step k_+, or not. If the latter, then, we finish with a set L'', containing at least $\frac{1}{2}(\lambda-1)k_+$ live vertices, and disjoint from L'. The chance (conditional on arriving at this stage) that there exists no edge between L' and L'' is bounded above by

$$(1-p)^{[\frac{1}{2}(\lambda-1)k_+]^2} \le \exp\left(-\tfrac{1}{4}\lambda(\lambda-1)^2 n^{1/3}\right) = \mathrm{o}(n^{-2}).$$

Therefore, the probability that there exist two distinct vertices belonging to distinct components of size exceeding k_+ is no greater than

$$1 - P\left(\bigcap_{v \in V} A_v\right) + n^2 \mathrm{o}(n^{-2}) = \mathrm{o}(1).$$

In summary, a.a.s., every component is either 'small' (smaller than k_-) or 'large' (larger than k_+), and there can be no more than one large component. In order to estimate the size of any such large component, we use Chebyshev's inequality to estimate the aggregate sizes of small components. Let $v \in V$. The chance $\sigma = \sigma(n, p)$ that v is in a small component satisfies

$$\eta_- - \mathrm{o}(1) \le \sigma \le \eta_+, \tag{11.3}$$

where η_+ (respectively, η_-) is the extinction probability of a branching process with family-size distribution $\mathrm{bin}(n - k_-, p)$ (respectively, $\mathrm{bin}(n, p)$), and the $\mathrm{o}(1)$ term bounds the probability that the latter branching process terminates after k_- or more steps. It is an easy exercise to show that $\eta_-, \eta_+ \to \eta$ as $n \to \infty$ where $\eta(\lambda) = 1 - \alpha(\lambda)$ is the extinction probability of a $\mathrm{Po}(\lambda)$ branching process.

The number S of vertices in small components satisfies

$$E(S) = \sigma n = (1 + \mathrm{o}(1))\eta n.$$

Furthermore, by an argument similar to that above,

$$E(S(S-1)) \le n\sigma\big[k_- + n\sigma(n - k_-, p)\big] = (1 + \mathrm{o}(1))(ES)^2,$$

whence, by Chebyshev's inequality, $G_{n,p}$ possesses $(\eta + \mathrm{o_p}(1))n$ vertices in small components. This leaves just $n - (\eta + \mathrm{o_p}(1))n = (\alpha + \mathrm{o_p}(1))n$ vertices remaining for the large component, and the theorem is proved. □

A further analysis yields the size X_n of the largest subcritical component, and the size Y_n of the second largest supercritical component.

(11.4) Theorem.

(a) *When* $\lambda < 1$,

$$X_n = (1 + \mathrm{o_p}(1)) \frac{\log n}{\lambda - 1 - \log \lambda}.$$

(b) *When* $\lambda > 1$,

$$Y_n = (1 + \mathrm{o_p}(1)) \frac{\log n}{\lambda' - 1 - \log \lambda'}$$

where $\lambda' = \lambda(1 - \alpha(\lambda))$.

If $\lambda > 1$, and we remove the largest component, we are left with a random graph on $n - X_n \sim n(1 - \alpha(\lambda))$ vertices. The mean vertex-degree of this subgraph is approximately

$$\frac{\lambda}{n} \cdot n(1 - \alpha(\lambda)) = \lambda(1 - \alpha(\lambda)) = \lambda'.$$

It may be checked that this is strictly smaller than 1, implying that the remaining subgraph behaves as a subcritical random graph on $n - X_n$ vertices. Theorem 11.4(b) now follows from part (a).

The picture is more interesting when $\lambda \approx 1$, for which there is a detailed combinatorial study of [128]. Rather than describing this here, we deviate to the work of David Aldous [16], who has demonstrated a link, via the multiplicative coalescent, to Brownian motion. We set

$$p = \frac{1}{n} + \frac{t}{n^{4/3}}$$

where $t \in \mathbb{R}$, and we write $C_n^t(1) \geq C_n^t(2) \geq \cdots$ for the component sizes of $G_{n,p}$ in decreasing order. We shall explore the weak limit (as $n \to \infty$) of the sequence $n^{-2/3}(C_n^t(1), C_n^t(2), \dots)$.

Let $W = (W(s) : s \geq 0)$ be a standard Brownian motion, and

$$W^t(s) = W(s) + ts - \tfrac{1}{2}s^2, \qquad s \geq 0,$$

a Brownian motion with drift $t - s$ at time s. Write

$$B^t(s) = W^t(s) - \inf_{0 \leq s' \leq s} W^t(s')$$

for a reflecting inhomogenous Brownian motion with drift.

(11.5) Theorem [16]. *As $n \to \infty$,*

$$n^{-2/3}(C_n^t(1), C_n^t(2), \dots) \Rightarrow (C^t(1), C^t(2), \dots),$$

where $C^t(j)$ is the length of the jth largest excursion of B^t.

We think of the sequences of Theorem 11.5 as being chosen at random from the space of decreasing non-negative sequences $\mathbf{x} = (x_1, x_2, \dots)$, with metric

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_i (x_i - y_i)^2}.$$

As t increases, two components of sizes x_i, x_j ‘coalesce’ at a rate proportional to the product $x_i x_j$. Theorem 11.5 identifies the scaling limit of this process as that of the evolving excursion-lengths of W^t reflected at zero. This observation has contributed to the construction of the so-called ‘multiplicative coalescent’.

In summary, the largest component of the subcritical random graph (when $\lambda < 1$) has order $\log n$, and of the supercritical graph (when $\lambda > 1$) order n. When $\lambda = 1$, the largest component has order $n^{2/3}$, with a multiplicative constant that is a random variable. The discontinuity at $\lambda = 1$ is sometimes referred to as the ‘Erdős–Rényi double jump’.

11.3 Independence and colouring

Our second random-graph exercise concerns the chromatic number of $G_{n,p}$ for constant p. The *chromatic number* $\chi(G)$ of a graph G is the least number of colours with the property that: there exists an allocation of colours to vertices such that no two neighbours have the same colour. Let $p \in (0, 1)$, and write $\chi_{n,p}$ for the chromatic number of $G_{n,p}$.

A subset W of V is called *independent* if no pair of vertices in W are adjacent, that is, if $X_{i,j} = 0$ for all $i, j \in W$. Any colouring of $G_{n,p}$ partitions V into independent sets, and therefore the chromatic number is related to the size $I_{n,p}$ of the largest independent set of $G_{n,p}$.

(11.6) Theorem [104]. *We have that*

$$I_{n,p} = (1 + \mathrm{o_p}(1)) 2 \log_\pi n.$$

where the base π of the logarithm is given by $\pi = 1/(1-p)$.

The proof follows a standard route: the upper bound follows by an estimate of an expectation, and the lower by an estimate of a second moment. When performed with greater care, such calculations yield much more accurate estimates of $I_{n,p}$ than those presented here, see, for example, [40], [129, Sect. 7.1], and [161, Sect. 2]. Specifically, there exists an integer-valued function $r = r(n, p)$ such that

$$P(r - 1 \le I_{n,p} \le r) \to 1 \qquad \text{as } n \to \infty. \tag{11.7}$$

Proof. Let N_k be the number of independent subsets of V with cardinality k. Then

$$P(I_{n,p} \ge k) = P(N_k \ge 1) \le E(N_k). \tag{11.8}$$

Now,

$$E(N_k) = \binom{n}{k}(1-p)^{\binom{k}{2}}, \tag{11.9}$$

With $\epsilon > 0$, set $k = 2(1+\epsilon)\log_\pi n$, and use the fact that

$$\binom{n}{k} \le \frac{n^k}{k!} \le (ne/k)^k,$$

to obtain

$$\log_\pi E(N_k) \le -(1 + \mathrm{o}(1)) k\epsilon \log_\pi n \to -\infty \qquad \text{as } n \to \infty.$$

By (11.8), $P(I_{n,p} \ge k) \to 0$ as $n \to \infty$. This is an example of the use of the so-called ‘first-moment method’.

A lower bound for $I_{n,p}$ is obtained by the 'second-moment method' as follows. By Chebyshev's inequality,

$$P(N_k = 0) \le P\big(|N_k - EN_k| \ge EN_k\big) \le \frac{\operatorname{var}(N_k)}{E(N_k)^2},$$

whence, since N_k takes values in the non-negative integers,

$$P(N_k \ge 1) \ge 2 - \frac{E(N_k^2)}{E(N_k)^2}. \tag{11.10}$$

Let $\epsilon > 0$ and $k = 2(1-\epsilon)\log_\pi n$. By (11.10), it suffices to show that

$$\frac{E(N_k^2)}{E(N_k)^2} \to 1 \qquad \text{as } n \to \infty. \tag{11.11}$$

By an elementary counting argument,

$$E(N_k^2) = \binom{n}{k}(1-p)^{\binom{k}{2}} \sum_{i=0}^{k} \binom{k}{i}\binom{n-k}{k-i}(1-p)^{\binom{k}{2}-\binom{i}{2}}.$$

After a minor analysis using (11.8) and (11.11), we deduce that $P(I_{n,p} \ge k) \to 1$ as $n \to \infty$. The theorem is proved. □

We turn now to the chromatic number $\chi_{n,p}$. Since the size of any set of vertices of given colour is no larger than $I_{n,p}$, one has trivially that

$$\chi_{n,p} \ge \frac{n}{I_{n,p}} = (1 + o_p(1))\frac{n}{2\log_\pi n}. \tag{11.12}$$

The sharpness of this inequality was proved by Béla Bollobás [39].

(11.13) Theorem [39]. *We have that*

$$\chi_{n,p} = (1 + o_p(1))\frac{n}{2\log_\pi n}.$$

The term $o_p(1)$ may be estimated quite precisely by a more detailed analysis than that presented here, see [39, 162] and [129, Sect. 7.3]. Specifically, one has, a.a.s., that

$$\chi_{n,p} = \frac{n}{2\log_\pi n - 2\log_\pi \log_\pi n + O_p(1)},$$

where $Z_n = O_p(y_n)$ means $P(|Z_n/y_n| > M) \le g(M) \to 0$ as $M \to \infty$.

Proof. The lower bound follows as in (11.12), and so we concentrate on finding an upper bound for $\chi_{n,p}$. Let $0 < \epsilon < \frac{1}{4}$, and write $k = \lfloor 2(1-\epsilon)\log_\pi n \rfloor$. We claim that, with probability $1 - o(1)$, *every* subset of V with cardinality at least

$m = \lfloor n/(\log_\pi n)^2 \rfloor$ possesses an independent subset of size at least k. The required bound on $\chi_{n,p}$ follows from this claim, as follows. We find an independent set of size k, and we colour its vertices with colour 1. From the remaining set of $n-k$ vertices, we find an independent set of size k, and we colour it with colour 2. This process may be iterated until there remains a set S of size smaller than $\lfloor n/(\log_\pi n)^2 \rfloor$. We colour the vertices of S 'greedily', with $|S|$ different colours. The total number of colours used is no larger than

$$\frac{n}{k} + \frac{n}{(\log_\pi n)^2},$$

which, for large n, is smaller than $\frac{1}{2}(1+2\epsilon)n/\log_\pi n$.

(11.14) Lemma. *The probability that $G_{m,p}$ contains no independent set of size k is less than* $\exp\bigl(-n^{\frac{7}{2}-2\epsilon+o(1)}/m^2\bigr)$.

There are $\binom{n}{m}$ ($< 2^n$) subsets of $\{1,2,\dots,n\}$ with cardinality m. The probability that some such subset fails to contain an independent set of size k is, by the lemma, no larger than

$$2^n \exp(-n^{\frac{7}{2}-2\epsilon+o(1)}/m^2) = o(1).$$

We turn to the proof of Lemma 11.14, for which we shall use the Hoeffding inequality, Theorem 4.18.

For $M \ge k$, we write

$$F(M,k) = \binom{M}{k}(1-p)^{\binom{k}{2}}.$$

We shall require M to be such that $F(M,k)$ grows in the manner of a power of n, and to that end we set

$$M = \lfloor (Ck/e)n^{1-\epsilon} \rfloor, \tag{11.15}$$

where

$$\log C = \frac{3\log\pi}{8(1-\epsilon)}$$

has been chosen in such a way that

$$F(M,k) = n^{\frac{7}{4}-\epsilon+o(1)}. \tag{11.16}$$

Let $\mathcal{I}(r)$ be the set of independent subsets of $\{1,2,\dots,r\}$ with size k. We write $N_k = |\mathcal{I}(m)|$, and N_k' for the number of elements I of $\mathcal{I}(m)$ with the property that $|I \cap I'| \le 1$ for all $I' \in \mathcal{I}(m)$, $I' \ne I$. Note that

$$N_k \ge N_k'. \tag{11.17}$$

We order as $(e_1, e_2, \dots, e_{\binom{m}{2}})$ the edges of the complete graph on the vertex-set $\{1, 2, \dots, m\}$. Let $\mathcal{F}_s$ be the σ-field generated by the states of the edges $e_1, e_2, \dots, e_s$, and let $Y_s = E(N'_k \mid \mathcal{F}_s)$. It is elementary that the sequence $(Y_s, \mathcal{F}_s)$, $0 \le s \le \binom{m}{2}$, is a martingale (see [109, Example 7.9.24]). The addition or removal of an edge may change N'_k by no more than 1, so the martingale differences satisfy $|Y_{s+1} - Y_s| \le 1$. Since $Y_0 = E(N'_k)$ and $Y_{\binom{m}{2}} = N'_k$,

$$(11.18)\qquad \begin{aligned} P(N_k = 0) &\le P(N'_k = 0)\\ &= P\big(N'_k - E(N'_k) \le -E(N'_k)\big)\\ &\le \exp\left(-\tfrac{1}{2}E(N'_k)^2 \Big/ \binom{m}{2}\right)\\ &\le \exp\big(-2E(N'_k)^2/m^2\big), \end{aligned}$$

by (11.17) and Theorem 4.18. We now require a lower bound for $E(N'_k)$.

Let M be as in (11.15). Let $M_k = |\mathcal{I}(M)|$, and let M'_k be the number of elements $I \in \mathcal{I}(M)$ such that $|I \cap I'| \le 1$ for all $I' \in \mathcal{I}(M)$, $I' \ne I$. Clearly

$$(11.19)\qquad N'_k \ge M'_k,$$

and we shall bound $E(M'_k)$ below. Let $K = \{1, 2, \dots, k\}$, and let A be the event that K is an independent set. Let Z be the number of elements of $\mathcal{I}(M)$, other than K, that intersect K in two or more vertices. Then

$$(11.20)\qquad \begin{aligned} E(M'_k) &= \binom{M}{k} P(A \cap \{Z = 0\})\\ &= \binom{M}{k} P(A) P(Z = 0 \mid A)\\ &= F(M, k) P(Z = 0 \mid A). \end{aligned}$$

We bound $P(Z = 0 \mid A)$ by

$$(11.21)\qquad \begin{aligned} P(Z = 0 \mid A) &= 1 - P(Z \ge 1 \mid A)\\ &\ge 1 - E(Z \mid A)\\ &= 1 - \sum_{t=2}^{k-1} \binom{k}{t}\binom{M-k}{k-t}(1-p)^{\binom{k}{2}-\binom{t}{2}}\\ &= 1 - \sum_{t=2}^{k-1} F_t, \qquad \text{say.} \end{aligned}$$

For $t \ge 2$,

$$(11.22)\qquad \begin{aligned} F_t/F_2 &= \frac{(M-2k+2)!}{(M-2k+t)!} \cdot \left(\frac{(k-2)!}{(k-t)!}\right)^2 \cdot \frac{2}{t!}(1-p)^{-\frac{1}{2}(t+1)(t-2)}\\ &\le \left[\frac{k^2(1-p)^{-\frac{1}{2}(t+1)}}{M-2k}\right]^{t-2}. \end{aligned}$$

For $2 \le t \le \frac{1}{2}k$,

$$\log_\pi\left[(1-p)^{-\frac{1}{2}(t+1)}\right] \le \tfrac{1}{4}(k+1) < \tfrac{1}{2} + \tfrac{1}{2}(1-\epsilon)\log_\pi n,$$

so $(1-p)^{-\frac{1}{2}(t+1)} = o(n^{\frac{1}{2}})$. By (11.22),

$$\sum_{2\le t\le \frac{1}{2}k} F_t = (1+o(1))F_2.$$

Similarly,

$$\begin{aligned} F_t/F_{k-1} &= \binom{k}{t}\binom{M-k}{k-t}\frac{(1-p)^{\frac{1}{2}(k+t-2)(k-t-1)}}{k(M-k)} \\ &\le \left[kn(1-p)^{\frac{1}{2}(k+t-2)}\right]^{k-t-1}. \end{aligned}$$

For $\frac{1}{2}k \le t \le k-1$, we have as above that

$$(1-p)^{\frac{1}{2}(k+t)} \le (1-p)^{\frac{3}{4}k} \le n^{-\frac{9}{8}}$$

whence

$$\sum_{\frac{1}{2}k<t\le k-1} F_t = (1+o(1))F_{k-1}.$$

In summary,

$$\sum_{t=2}^{k-1} F_t = (1+o(1))(F_2 + F_{k-1}). \tag{11.23}$$

By (11.16),

$$\begin{aligned} F_2 &\le \frac{k^4}{2(1-p)(M-k)^2}F(M,k) \\ &= n^{-\frac{1}{4}+\epsilon+o(1)} = o(1), \end{aligned}$$

and similarly

$$F_{k-1} = k(M-k)(1-p)^{k-1} = o(1).$$

By (11.23) and (11.20)–(11.21),

$$E(M_k') = (1+o(1))F(M,k) = n^{\frac{7}{4}-\epsilon+o(1)}.$$

Returning to the martingale bound (11.18), it follows by (11.19) that

$$P(N_k = 0) \le \exp(-n^{\frac{7}{2}-2\epsilon+o(1)}/m^2)$$

as required. □

11.4 Exercises

11.1. Let $\eta(\lambda)$ be the extinction probability of a branching process whose family-sizes are distributed as $\text{Po}(\lambda)$. Show that

$$\eta(\lambda) = \frac{1}{\lambda} \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} (\lambda e^{-\lambda})^k.$$

11.2. [129] *Chernoff bounds.* Let X be distributed as $\text{bin}(n, p)$ and $\lambda = np$. Show that

$$P(X \geq E(X) + t) \leq e^{-\lambda\phi(t/\lambda)} \leq \exp\left(-\frac{t^2}{2(\lambda + t/3)}\right), \quad t \geq 0,$$

$$P(X \leq E(X) - t) \leq e^{-\lambda\phi(-t/\lambda)} \leq \exp\left(-\frac{t^2}{\lambda}\right), \quad t \geq 0,$$

where

$$\phi(x) = \begin{cases} (1 + x)\log(1 + x) - x & \text{if } x \geq -1, \\ \infty & \text{if } x < -1. \end{cases}$$

11.3. Consider a branching process with family-size distribution $\text{bin}(n, \lambda/n)$. Show that the extinction probability converges to $\eta(\lambda)$ as $n \to \infty$, where $\eta(\lambda)$ is the extinction probability of a $\text{Po}(\lambda)$ branching process.

11.4. [40] Show that the size of the largest independent set of $G_{n,p}$ is, a.a.s., either $r - 1$ or r, for some deterministic function $r = r(n, p)$.

12

Lorentz Gas

A small particle is fired through an environment of large particles, and is subjected to reflections on impact. Little is known about the trajectory of the small particle when the larger ones are distributed at random. The notorious problem on the square lattice is summarized, and open questions are posed for the case of a continuum of needle-like mirrors in the plane.

12.1 Lorentz model

In a famous sequence [154] of papers of 1906, Hendrik Lorentz introduced the following problem. Large (heavy) particles are distributed about $\mathbb{R}^d$. A small (light) particle is fired through $\mathbb{R}^d$, with a trajectory comprising straight-line segments between the points of interaction with the heavy particles. When the small particle hits a heavy particle, the small particle is reflected, and the large particle remains motionless. See Figure 12.1 for an illustration.

We may think of the heavy particles as objects bounded by reflecting surfaces, and the light particle as a photon. The problem is to say something non-trivial about how the trajectory of the photon depends on the 'environment' of heavy particles. Conditional on the enviroment, the photon pursues a deterministic path about which the natural questions include:

1. is the path unbounded?
2. how distant is the photon from its starting point after the elapse of time t?

For simplicity, we assume henceforth that the large particles are identical to one another, and the small particle is of negligible volume.

Probability may be injected naturally into this model through the assumption that the heavy particles are distributed at random around $\mathbb{R}^d$ according to some probability measure μ. The questions above may be rephrased, and made more precise, in the language of probability theory. Let X_t denote the position of the photon at time t, assuming constant velocity. Under what conditions on μ:

I. is there strictly positive probability that the function X_t is unbounded?

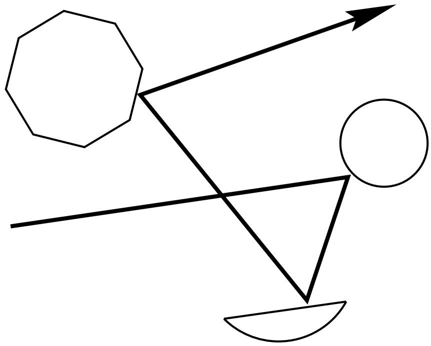

Figure 12.1. The trajectory of the photon comprises straight-line segments between the points of reflection.

II. does X_t converge to a Brownian motion, after suitable re-scaling?

For a wide choice of measures μ, these questions are currently unanswered.

The Lorentz gas is very challenging to mathematicians, and little is known rigorously in answer to the questions above. The reason is that, as the photon moves around space, it gathers information about the random environment, and it carries this information with it for ever more.

The Lorentz gas was developed by Paul Ehrenfest [74]. For the relevant references in the mathematics and physics journals, the reader is referred to [95, 96]. Many references may be found in [202].

12.2 The square Lorentz gas

Probably the most provocative version of the Lorentz gas for probabilists arises when the light ray is confined to the square lattice $\mathbb{L}^2$. At each vertex v of $\mathbb{L}^2$, we place a 'reflector' with probability p, and nothing otherwise (the occupancies of different vertices are independent). Reflectors come in two types: 'NW' and 'NE'. A NW reflector deflects incoming rays heading northwards (respectively, southwards) to the west (respectively, east) and vice versa. NE reflectors behave similarly with east and west interchanged. See Figure 12.2. Think of a reflector as being a two-sided mirror placed at 45° to the axes, so that an incoming light ray is reflected along an axis perpendicular to its direction of arrival. Now, for each vertex x, with probability p we place a reflector at x, and otherwise we place nothing at x. This is done independently for different x. If a reflector is placed at x, then we specify that it is equally likely to be NW as NE.

We shine a torch northwards from the origin. The light is reflected by the mirrors, and we ask whether or not the light ray returns to the origin. Letting

$$\eta(p) = P_p(\text{the light ray returns to the origin}),$$

we ask when is it the the case that $\eta(p) = 1$. It is reasonable to conjecture that η

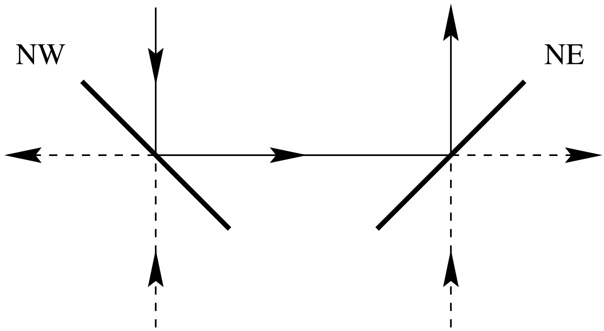

Figure 12.2. An illustration of the effects of NW and NE reflectors on the light ray.

is non-decreasing in p. Certainly $\eta(0) = 0$, and it is well known[1] that $\eta(1) = 1$.

(12.1) Theorem. *We have that* $\eta(1) = 1$.

We invite the reader to consider whether or not $\eta(p) = 1$ for some $p \in (0, 1)$. A variety of related conjectures, not entirely self-consistent, may be found in the physics literature. There are almost no mathematical results about this process beyond Theorem 12.1. We mention the paper [179], where it is proved that the number $N(p)$ of unbounded light rays on $\mathbb{Z}^2$ is almost surely constant, and is equal to one of 0, 1, ∞. Furthermore, if there exist unbounded light trajectories, then they self-intersect infinitely often. If $N(p) = \infty$, the position X_n of the photon at time n, when following an unbounded trajectory, is superdiffusive in the sense that $E(|X_n|^2)/n$ is unbounded as $n \to \infty$. The principal method of [179] is to observe the environment of mirrors as viewed from the moving photon.

In a variant of the standard random walk, termed the 'burn-your-bridges' random walk by Omer Angel, an edge is destroyed immediately after it is traversed for the first time. When $p = \frac{1}{3}$, the photon follows a burn-your-bridges random walk on $\mathbb{L}^2$.

Proof. We construct an ancillary lattice $\mathcal{L}$ as follows. Let

$$A = \left\{(m + \tfrac{1}{2}, n + \tfrac{1}{2}) : m + n \text{ is even}\right\}.$$

Let $\sim$ be the adjacency relation on A given by $(m + \frac{1}{2}, n + \frac{1}{2}) \sim (r + \frac{1}{2}, s + \frac{1}{2})$ if and only if $|m - r| = |n - s| = 1$. We obtain thus a graph $\mathcal{L}$ on A that is isomorphic to $\mathbb{L}^2$. See Figure 12.3.

We declare the edge of $\mathcal{L}$ joining $(m - \frac{1}{2}, n - \frac{1}{2})$ to $(m + \frac{1}{2}, n + \frac{1}{2})$ to be *open* if there is a NE mirror at (m, n); similarly we declare the edge joining $(m - \frac{1}{2}, n + \frac{1}{2})$ to $(m + \frac{1}{2}, n - \frac{1}{2})$ to be *open* if there is a NW mirror at (m, n). Edges that are not open are designated *closed*. This defines a bond percolation process in which north-easterly and north-westerly edges are open with probability $\frac{1}{2}$. Since $p_c(\mathbb{L}^2) = \frac{1}{2}$, the process is critical. See Section 5.5.

[1] See the historical remark in [94].

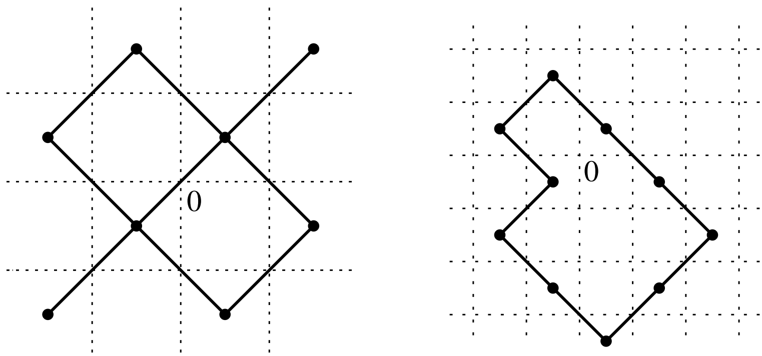

Figure 12.3. (a) The heavy lines form the lattice $\mathcal{L}$, and the central point is the origin of $\mathbb{L}^2$. (b) An open circuit in $\mathcal{L}$ constitutes a barrier of mirrors through which no light may penetrate.

Let N be the number of open circuits in $\mathcal{L}$ with the origin in their interiors. By the remarks prior to Theorem 5.33, we have that $\mathbb{P}(N \geq 1) = 1$, where $\mathbb{P}$ is the appropriate probability measure. Such an open circuit corresponds to a barrier of mirrors surrounding the origin (see Figure 12.3), from which no light can escape. Therefore $\eta(1) = 1$. □

The problem above may be stated for other lattices such as $\mathbb{L}^d$, see [94] for example. It is much simplified if one allows the photon to flip its own coin as it proceeds through the disordered medium of mirrors. Two such models have been explored. In the first, there is a positive probability that the photon will misbehave when it hits a mirror, see [207]. In the second, there is allowed a small density of vertices at which the photon acts in the manner of a random walk, see [35, 105].

12.3 In the plane

Here is a continuum version of the Lorentz gas. Let Π be a Poisson process in $\mathbb{R}^2$ with intensity 1. For each $x \in \Pi$, we place a needle (that is, a closed rectilinear line-segment) of given length l with its centre at x. The orientations of the needles are taken to be independent random variables with a common law μ on $[0, \pi)$. We call μ *degenerate* if it has support on a singleton, that is, all needles are (almost surely) parallel.

Each needle is interpreted as a (two-sided) reflector of light. Needles are permitted to overlap. Light is projected from the origin northwards, and deflected by the needles. Since the light strikes the endpoint of some needle with probability 0, we shall overlook this possibility.

In a related problem, we may study the union M of the needles, viewed as subsets of $\mathbb{R}^2$, and ask whether either (or both) of the sets M, $\mathbb{R}^2 \setminus M$ contains an unbounded component. This problem is known as ‘needle percolation’, and it has received some attention (see, for example, [163, Sect. 8.5], and also [120]). Of concern to us in the present setting is the following. Let $\lambda(l)$ be the probability that there exists an unbounded path of $\mathbb{R}^2 \setminus M$ with the origin 0 as endpoint. It is clear that $\lambda(l)$ is non-increasing in l. The following is a fairly straightforward exercise of percolation type.

(12.2) Theorem [120]. *There exists $l_c = l_c(\mu) \in (0, \infty]$ such that*

$$\lambda(l) \begin{cases} > 0 & \text{if } l < l_c, \\ = 0 & \text{if } l > l_c, \end{cases}$$

and furthermore $l_c < \infty$ if and only if μ is non-degenerate.

The phase transition has been defined here in terms of the existence of an unbounded 'vacant path' from the origin. When no such path exists, the origin is almost surely surrounded by a cycle of pairwise-intersecting needles. That is,

$$\mathbb{P}_\mu(E) \begin{cases} < 1 & \text{if } l < l_c, \\ = 1 & \text{if } l > l_c, \end{cases} \tag{12.3}$$

where E is the event that there exists a component C of needles such that the origin of $\mathbb{R}^2$ lies in a bounded component of $\mathbb{R}^2 \setminus C$, and $\mathbb{P}_\mu$ denotes the probability measure governing the configuration of mirrors.

The needle percolation problem is a type of continuum percolation model, cf. the space–time percolation process of Section 6.6. Continuum percolation, and in particular the needle (or 'stick') model, has been summarized in [163, Sect. 8.5].

We return to the above Lorentz problem. Suppose that the photon is projected from the origin at angle θ to the x-axis, for given $\theta \in [0, 2\pi)$. Let Θ be the set of all θ such that the trajectory of the photon is unbounded. It is clear from Theorem 12.2 that $\mathbb{P}_\mu(\Theta = \varnothing) = 1$ if $l > l_c$. The strength of the following theorem of Matthew Harris lies in the converse statement.

(12.4) Theorem [120]. *Let μ be non-degenerate, with support a subset of the rational angles $\pi\mathbb{Q}$.*

(a) *If $l > l_c$, then $\mathbb{P}_\mu(\Theta = \varnothing) = 1$.*
(b) *If $l < l_c$, then*

$$\mathbb{P}(\Theta \text{ has Lebesgue measure } 2\pi) = 1 - \mathbb{P}_\mu(E) > 0.$$

That is to say, almost surely on the complement of E, the set Θ differs from the entire interval $[0, 2\pi)$ by a null set. The proof uses a type of dimension-reduction method, and utilizes a theorem concerning so-called 'interval-exchange transformations' taken from ergodic theory, see [134]. It is a key assumption for this argument that μ be supported within the rational angles.

Theorem 12.4 leaves open even the arguably most natural instance of the problem, in which μ is uniform on $[0, \pi)$. Let $\eta(l)$ be the probability that the light ray is bounded, having started by heading northwards from the origin. As above,

$\eta(l) = 1$ when $l > l_c$. In contrast, it is not known for general μ whether or not $\eta(l) < 1$ for sufficiently small positive l. It seems reasonable to conjecture the following. For any probability measure μ on $[0, \pi)$, there exists $l_r \in (0, l_c]$ such that $\eta(l) < 1$ whenever $l < l_r$. This conjecture is open even for the case when μ is uniform on $[0, \pi)$.

(12.5) Conjecture. *Let μ be the uniform probability measure on $[0, \pi)$, and let l_c denote the critical length for the associated needle percolation problem (as in Theorem 12.2).*

(a) *There exists $l_r \in (0, l_c]$ such that*

$$\eta(l) \begin{cases} < 1 & \text{if } l < l_r, \\ = 1 & \text{if } l > l_r, \end{cases}$$

(b) *We have that $l_r = l_c$.*

As a first step, we seek a proof that $\eta(l) < 1$ for sufficiently small positive values of l. It is typical of such mirror problems that we lack even a proof that $\eta(l)$ is monotone in l.

12.4 Exercises

12.1. There are two ways of putting in the barriers in the percolation proof of Theorem 12.1, depending on whether one uses the odd or the even vertices. Use this fact to establish bounds for the tail of the size of the trajectory when the density of mirrors is 1.

12.2. In a variant of the square Lorentz lattice gas, NE mirrors occur with probability η and NW mirrors otherwise. Show that the photon's trajectory is almost surely bounded.

12.3. Needles are dropped in the plane in the manner of a Poisson process with intensity 1. They have length l, and their angles to the horizontal are independent random variables with law μ. Show that there exists $l_c = l_c(\mu) \in (0, \infty]$ such that: the probability that the origin lies in an unbounded path intersecting no needle is strictly positive when $l < l_c$, and equals zero when $l > l_c$.

12.4. (continuation) Show that $l_c < \infty$ if and only if μ is non-degenerate.

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